Linear Algebra with Application (LAWA 2020)

Homework 1

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Corollary 2.23 *Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$. If A and AB are both invertible, then B is also invertible.*

Proof. Let A^{-1} and $(AB)^{-1}$ be the inverses of A and AB respectively. Note that AB is a $n \times n$ matrix, as well as the inverses A^{-1} and $(AB)^{-1}$. Recall from point 3. of Theorem 2.22, that $(AB)^{-1} = B^{-1}A^{-1}$.

Let us consider the matrix $C = (AB)^{-1}A$. We claim that $C = B^{-1}$. Indeed

$$CB = (AB)^{-1}A\,B = (B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

and

$$BC = B(AB)^{-1}A = B(B^{-1}A^{-1})A = (B^{-1}B)(A^{-1}A) = II = I,$$

which proves the claim and thus the result. ∎

Proposition 2.26 *Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$ be two diagonal matrices. Then*

1. *$A + B$ is diagonal;*

2. AB is diagonal.

Proof. Let $A = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ and $B = \operatorname{diag}(b_{11}, b_{22}, \ldots, b_{nn})$, that is

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}$$

for certain numbers $a_{11}, a_{22}, \ldots, a_{nn}, b_{11}, b_{22}, \ldots, b_{nn} \in \mathbb{R}$.

Then one has $A + B = \operatorname{diag}(a_{11} + b_{11}, a_{22} + b_{22}, \ldots, a_{nn} + b_{nn})$, that is

$$A + B = \begin{pmatrix} a_{11} + b_{11} & 0 & \cdots & 0 \\ 0 & a_{22} + b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} + b_{nn} \end{pmatrix}$$

and $A \cdot B = \operatorname{diag}(a_{11}b_{11}, a_{22}b_{22}, \ldots, a_{nn}b_{nn})$, that is

$$AB = \begin{pmatrix} a_{11}b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}b_{nn} \end{pmatrix}.$$

Another way of proving it is by noticing that, if we denote $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$, then we have $A + B = \left(c_{ij}\right)$ and $AB = \left(d_{ij}\right)$ with

$$c_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad d_{ij} = \sum_{k=1}^{n} a_{jk} b_{kj}$$

for all i, j. Since A and B are diagonal, we have $a_{ij} = 0$ for all $i \neq j$ and $b_{ij} = 0$ for all $i \neq j$. Thus for all $i \neq j$ one has

$$c_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0$$

and

$$\begin{aligned} d_{ij} &= \sum_{k \neq i,j}^{n} a_{ik} b_{kj} \ + \ a_{ii} b_{ij} \ + \ a_{ij} b_{jj} \\ &= \sum_{k \neq i,j}^{n} 0 \cdot 0 \ + \ a_{ii} \cdot 0 \ + \ 0 \cdot b_{jj} \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

∎

Proposition 2.28 *Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$ be two upper triangular matrices. Then*

1. $A + B$ *is upper triangular;*

2. AB *is upper triangular.*

Proof. Let $A = \big(a_{ij}\big)$ and $B = \big(b_{ij}\big)$. Since the two matrices are upper triangular, then we have $a_{ij} = b_{ij} = 0$ for all $j < i$, that is the two matrices are of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}$$

for certain numbers a_{ij}, b_{ij} with $1 \leq j < i \leq n$.

Then we have $A + B = \big(c_{ij}\big)$ and $AB = \big(d_{ij}\big)$ with

$$c_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad d_{ij} = \sum_{k=1}^{n} a_{jk} b_{kj}$$

for all i, j. Thus, for all $j < i$, we have

$$c_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0$$

and

$$\begin{aligned} d_{ij} &= \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^{n} a_{ik} b_{kj} \\ &= \sum_{k=1}^{i-1} 0 \cdot b_{kj} + \sum_{k=i}^{n} a_{ik} \cdot 0 \\ &= 0 + 0 = 0 \end{aligned}$$

where the second equality holds because $a_{ik} = 0$ for all $k \leq i - 1 < i$ and $b_{kj} = 0$ for all $j < i \leq k$. ■

Example 3.4 The system

$$\begin{cases} x + y &= 1 \\ 2x + 2y &= 3 \end{cases}$$

has no solution, so it is inconsistent. Indeed, from trom the first equation we find that

$$y = 1 - x.$$

If we substitute y with $1 - x$ in the second equation, we find

$$2x + 2(1 - x) = 2x + 2 - 2x = 3$$

that is

$$2 = 3$$

which gives us a contradiction.

Example 3.5 The system

$$\begin{cases} x + y + z &= 2 \\ x - y + z &= 0 \end{cases}$$

has infinitely many solutions. Indeed, from trom the first equation we find that

$$z = 2 - x - y.$$

If we substitute z with $2 - x - y$ in the second equation, we find

$$x - y + 2 - x - y = 2 - 2y = 2(1 - y) = 0,$$

which implies that $y = 1$. Thus we obtain the equation on two variables

$$x + z = 1,$$

which hold for every $z = 1 - x$. Hence, all solutions have the form

$$X = \begin{pmatrix} s & 1 & 1 - s \end{pmatrix}$$

for a parameter $s \in \mathbb{R}$.