Linear Algebra with Application (LAWA 2020)

Homework 2

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Exercise 1 [Equation (3.11)]

Let us consider the matrix

$$\begin{pmatrix} 1 & -4 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The matrix is already in row-echelon form. In order to find its reduced row-echelon equivalent we need to *clean* the column containing the leading 1s.

The first leading one, in position $(1,1)$ is the only non-zero element of the column. Above the second leading one, in position $(2,3)$, there is a non-zero element, that is 1 (in position $(1,3)$). To obtain a zero in its place, we subtract the second row from the first (elementary row operation of type *iii*), and we obtain

$$\begin{pmatrix} 1 & -4 & 0 & 3 & 6 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Now we do the same operation on the third column, containing the third leading one, that is we subtract 3 times the third row to the first one, obtaining the matrix

$$\begin{pmatrix} 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

and then we add 3 times the third row to the second one, finally finding

$$\begin{pmatrix} 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

that is in reduced row-echelon form.

Exercise 2 [Example 3.18]

Let us find all solutions of the following system of linear equations

$$\begin{cases} x + y - z &= 3 \\ -2x - y &= -4 \\ 4x + 2y + 3z &= -1 \end{cases}$$

The augmented matrix of the system is

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ -2 & -1 & 0 & -4 \\ 4 & 2 & 3 & -1 \end{pmatrix}$$

The reduction of the augmented matrix (to its reduced row-echelon form) is

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ -2 & -1 & 0 & -4 \\ 4 & 2 & 3 & -1 \end{pmatrix} \xrightarrow[R_2 \to R_2 + 2R_1]{iii)} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -2 & 2 \\ 4 & 2 & 3 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 4R_1]{iii)} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & -2 & 7 & -13 \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 - R_2]{iii)} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & -2 & 7 & -13 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 2R_2]{iii)} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{pmatrix}$$

$$\xrightarrow[R_3 \to \frac{1}{2}R_3]{ii)} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 - R_3]{iii)} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 + 2R_3]{iii)} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix}.$$

The last matrix is the augmented matrix of the system of linear equations

$$\begin{cases} x &= 4 \\ y &= -4 \\ z &= -3 \end{cases}$$

which has solution

$$X = \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 4 & -4 & -3 \end{pmatrix}.$$

Since the two systems are equivalent, then X is also a solution (actually the unique solution) of the original system.

Exercise 3 [Example 3.15]

Let us consider the two augmented matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first one represent a system of 3 linear equations in 3 variables and the second a system of 5 linear equations in 5 variables.

The two matrices, which are in (reduced) row-echelon form, have both 3 leading ones, that is

$$\operatorname{rank}(A) = 3 \quad \text{and} \quad \operatorname{rank}(B) = 3.$$

Since the rank of A is equal to the number of variables of the first associated system, it follows from Theorem 3.19 that the solution is unique. On the other hand, since the rank of B is strictly less than the number of variables of the second system, and using again Theorem 3.19, we have that the set of solutions is infinite, and that it has $2(= 5 - 3)$ parameters.