Linear Algebra with Application (LAWA 2020)

Homework 4

Francesco Dolce

francesco.dolce@fjfi.cvut.cz

April 2020

Exercise 1 [Example 4.2]

Let us consider the two matrices

$$A = \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

with $x_1, x_2, x_3, x_4 \in \mathbb{R}$ and let us suppose that $AB = I$. Thus, we have

$$\begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - 5x_3 & 2x_2 - 5x_4 \\ x_1 + 2x_3 & x_2 + 2x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This give us the following system of four linear equations in four variables

$$\begin{cases} 2x_1 - 5x_3 & = & 1 \\ 2x_2 - 5x_4 & = & 0 \\ x_1 + 2x_3 & = & 0 \\ x_2 + 2x_4 & = & 1 \end{cases}.$$

To find the solution of this system we use the Gaussian algorithm on the augmented matrix of the system and get the equivalent reduced row-echelon matrix, as follows

$$\begin{pmatrix}2 & 0 & -5 & 0 & 1\\ 0 & 2 & 0 & -5 & 0\\ 1 & 0 & 2 & 0 & 0\\ 0 & 1 & 0 & 2 & 1\end{pmatrix} \xrightarrow[\substack{R_1 \leftrightarrow R_3\\ R_2 \leftrightarrow R_4}]{i)} \begin{pmatrix}1 & 0 & 2 & 0 & 0\\ 0 & 1 & 0 & 2 & 1\\ 2 & 0 & -5 & 0 & 1\\ 0 & 2 & 0 & -5 & 0\end{pmatrix}$$

$$\xrightarrow[\substack{R_3 \to R_3 - 2R_1\\ R_4 \to R_4 - 2R_2}]{iii)} \begin{pmatrix}1 & 0 & 2 & 0 & 0\\ 0 & 1 & 0 & 2 & 1\\ 0 & 0 & -9 & 0 & 1\\ 0 & 0 & 0 & -9 & -2\end{pmatrix}$$

$$\xrightarrow[\substack{R_3 \to -\frac{1}{9}R_3\\ R_4 \to -\frac{1}{9}R_4}]{ii)} \begin{pmatrix}1 & 0 & 2 & 0 & 0\\ 0 & 1 & 0 & 2 & 1\\ 0 & 0 & 1 & 0 & -\frac{1}{9}\\ 0 & 0 & 0 & 1 & \frac{2}{9}\end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 - 2R_3\\ R_2 \to R_2 - 2R_4}]{iii)} \begin{pmatrix}1 & 0 & 0 & 0 & \frac{2}{9}\\ 0 & 1 & 0 & 0 & \frac{5}{9}\\ 0 & 0 & 1 & 0 & -\frac{1}{9}\\ 0 & 0 & 0 & 1 & \frac{2}{9}\end{pmatrix}$$

Thus we get

$$\begin{pmatrix}x_1 & x_2 & x_3 & x_4\end{pmatrix} = \begin{pmatrix}\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} & \frac{2}{9}\end{pmatrix}.$$

Exercise 2 [Example 4.6]

Let us consider the matrix

$$A = \begin{pmatrix}1 & 2 & -1\\ 2 & 3 & -5\\ 4 & 1 & 1\end{pmatrix}.$$

To find the inverse of A let us use the matrix inversion algorithm, that is let us find the reduction into a reduced row-echelon form of the matrix $\begin{pmatrix}A & I\end{pmatrix}$

$$\begin{pmatrix}1 & 2 & -1 & 1 & 0 & 0\\ 2 & 3 & -5 & 0 & 1 & 0\\ 4 & 1 & 1 & 0 & 0 & 1\end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - 4R_1}]{iii)} \begin{pmatrix}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -1 & -3 & -2 & 1 & 0\\ 0 & -7 & 5 & -4 & 0 & 1\end{pmatrix}$$

$$\xrightarrow[R_2 \to -R_2]{ii)} \begin{pmatrix}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & 3 & 2 & -1 & 0\\ 0 & -7 & 5 & -4 & 0 & 1\end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 - 2R_2\\ R_3 \to R_3 + 7R_1}]{iii)} \begin{pmatrix}1 & 0 & -7 & -3 & 2 & 0\\ 0 & 1 & 3 & 2 & -1 & 0\\ 0 & 0 & 26 & 10 & -7 & 1\end{pmatrix}$$

$$\xrightarrow[R_3 \to \frac{1}{26} R_3]{ii)} \begin{pmatrix} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{5}{13} & -\frac{7}{26} & \frac{1}{26} \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 + 7R_2 \\ R_2 \to R_2 - 3R_3}]{iii)} \begin{pmatrix} 1 & 0 & 0 & -\frac{4}{13} & \frac{3}{26} & \frac{7}{26} \\ 0 & 1 & 0 & \frac{11}{13} & -\frac{5}{26} & -\frac{3}{26} \\ 0 & 0 & 1 & \frac{5}{13} & -\frac{7}{26} & \frac{1}{26} \end{pmatrix}$$

Thus the inverse of A is the matrix

$$A^{-1} = \begin{pmatrix} -\frac{4}{13} & \frac{3}{26} & \frac{7}{26} \\ \frac{11}{13} & -\frac{5}{26} & -\frac{3}{26} \\ \frac{5}{13} & -\frac{7}{26} & \frac{1}{26} \end{pmatrix} = \frac{1}{26} \begin{pmatrix} -8 & 3 & 7 \\ 22 & -5 & -3 \\ 10 & -7 & 1 \end{pmatrix}.$$

Exercise 3 [Example 4.14]

Let A, B be two square matrices and let us suppose that $A^3 = B$ and that B is invertible. Then, using Theorem 4.12, we can prove that A is invertible too.

Indeed, since B is invertible, then B^{-1} exists. Let us multiply B^{-1} to both sides of $A^3 = B$. We get

$$B^{-1}A^3 = B^{-1}B = I \qquad \text{and} \qquad A^3B^{-1} = BB^{-1} = I.$$

Thus $(B^{-1}A^2)A = I$. By Theorem 4.12 we know that $A(B^{-1}A^2) = I$ too. This proves that A is invertible, and its inverse is $A^{-1} = B^{-1}A^2$.