Linear Algebra with Application (LAWA 2020)

Homework 5

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Exercise 1 [Example 4.18]

Let us consider the three elementary matrices

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

As seen in Example 4.16, the elementary row operation that produces E_1 from I is

$$I \xrightarrow[R_1 \leftrightarrow R_3]{i)} E_1.$$

To obtain the matrix E_1^{-1} we consider the inverse of this operation, that is the (same) operation

$$I \xrightarrow[R_3 \leftrightarrow R_1]{i)} E_1^{-1}$$

and thus the inverse of E_1 is the (same) matrix

$$E_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The elementary row operation that produces E_2 from I is

$$I \xrightarrow[R_2 \to \frac{1}{3}R_2]{ii)} E_2.$$

To obtain the matrix E_2^{-1} we consider the inverse of this operation, that is the operation

$$I \xrightarrow[R_2 \to 3R_2]{ii)} E_2^{-1}$$

and thus the inverse of E_2 is the matrix

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The elementary row operation that produces E_3 from I is

$$I \xrightarrow[R_3 \to R_3 - 2R_1]{iii)} E_3.$$

To obtain the matrix E_3^{-1} we consider the inverse of this operation, that is the operation

$$I \xrightarrow[R_3 \to R_3 + 2R_1]{iii)} E_3^{-1}$$

and thus the inverse of E_3 is the matrix

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

To double check one can easy verify that

$$E_1E_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = E_1^{-1}E_1,$$

$$E_2E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_2^{-1}E_2$$

and

$$E_3E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = E_3^{-1}E_3.$$

Exercise 2 [Example 4.23]

Let us consider the matrix

$$A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & -1 & 0 \end{pmatrix}.$$

A possible reduction of A to an equivalent matrix B in reduced row-echelon form is the following:

$$\begin{aligned} A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & -1 & 0 \end{pmatrix} & \xrightarrow[R_1 \leftrightarrow R2]{i)} \begin{pmatrix} 1 & -1 & 0 \\ 3 & -2 & 5 \end{pmatrix} \\ & \xrightarrow[R_2 \to R2 - 3R_1]{iii)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 5 \end{pmatrix} \\ & \xrightarrow[R_1 \to R_1 + R2]{iii)} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \end{pmatrix} = B \end{aligned}$$

The elementary matrices corresponding to the previous elementary operations are, in order:

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We thus have $B = UA$, where

$$U = E_3 E_2 E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -3 \end{pmatrix}.$$

Exercise 3 [Example 4.27]

Let us consider the matrix

$$A = \begin{pmatrix} 3 & -3 & 6 \\ 1 & -1 & 1 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R}).$$

Let us use Theorem 4.25 to show that there exist two matrices U, V such that

$$UAV = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$$

with $r = \text{rank}\,(A)$. Let us first consider the reduction $\begin{pmatrix} A & I_2 \end{pmatrix} \to \begin{pmatrix} R & U \end{pmatrix}$ with R in reduced row-echelon form.

$$\begin{aligned} \begin{pmatrix} 3 & -3 & 6 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 \end{pmatrix} & \xrightarrow[R_1 \leftrightarrow R_2]{i)} \begin{pmatrix} 1 & -1 & 1 & 0 & 1 \\ 3 & -3 & 6 & 1 & 0 \end{pmatrix} \\ & \xrightarrow[R_2 \to R_2 - 3R_1]{iii)} \begin{pmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 1 & -3 \end{pmatrix} \\ & \xrightarrow[R_2 \to \frac{1}{3} R_2]{ii)} \begin{pmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -1 \end{pmatrix} \end{aligned}$$

$$\xrightarrow[R_1 \to R_1 - R_2]{iii)} \left(\begin{array}{ccc|cc} 1 & -1 & 0 & -\frac{1}{3} & 2 \\ 0 & 0 & 1 & \frac{1}{3} & -1 \end{array}\right).$$

Thus we have

$$R = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} -\frac{1}{3} & 2 \\ \frac{1}{3} & -1 \end{pmatrix}.$$

Moreover, since R has two leadings ones, we have $\operatorname{rank}(A) = \operatorname{rank}(R) = 2$. Using the second step of Theorem 4.25, we obtain:

$$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_3]{i)} \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + R_1]{iii)} \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

$$= \left(\begin{pmatrix} I_2 \\ O_{1,2} \end{pmatrix} \quad V^T \right).$$

where

$$V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Finally, one can check that we actually have

$$\begin{pmatrix} -\frac{1}{3} & 2 \\ \frac{1}{3} & -1 \end{pmatrix} \begin{pmatrix} 3 & -3 & 6 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

that is

$$UAV = \begin{pmatrix} I_2 & O_{2,1} \end{pmatrix}.$$