Linear Algebra with Application (LAWA 2020)

Homework 6

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May 2020

Exercise 1 [Example 5.3]

Let us consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & 2 \\ 5 & 0 & 3 \end{pmatrix}.$$

To compute the determinant of A we use the Laplace expansion along the first row:

$$\det(A) = 1C_{11}(A) - 2C_{12}(A) + 0C_{13}(A).$$

The $(1,1)$-cofactor of A is given by

$$C_{11}(A) = (-1)^{1+1} \det \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = +(\cdot 1 \cdot 3 - 2 \cdot 0) = 3,$$

while the $(1,3)$-cofactor of A is

$$C_{12}(A) = (-1)^{1+2} \det \begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix} = -(-1 \cdot 3 - 2 \cdot 5) = 13.$$

Since the $(1,3)$-entry of A is zero, we don't need to compute the $(1,3)$-cofactor of A. But if you're curious about it, here it is:

$$C_{13}(A) = (-1)^{1+3} \det \begin{pmatrix} -1 & 1 \\ 5 & 0 \end{pmatrix} = +(-1 \cdot 0 - 1 \cdot 5) = 5.$$

Thus, the determinat of A is

$$\det(A) = 1 \cdot 3 - 2 \cdot 13 + 0 \cdot 5 = -23.$$

Exercise 2 [Example 5.11]

Let us consider the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 4 & 7 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{pmatrix}.$$

The determinant of the first matrix is:

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \\ &= 0 \end{aligned}$$

where the first two equalities come from the fact that we used two elementary row operations of type *iii*): subtracting twice the 1-row from the 3-row and subtracting the 2-row from the 3-row.

The determinant of the second matrix is:

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\begin{aligned}
\det(B) &= \det\begin{pmatrix} 1-a & a-1 & 0 & 0 \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{pmatrix} && R_1 \to R_1 - R_2 \\
&= \det\begin{pmatrix} 1-a & 0 & 0 & 0 \\ a & a+1 & a & a \\ a & 2a & 1 & a \\ a & 2a & a & 1 \end{pmatrix} && C_2 \to C_2 + C_1 \\
&= (1-a)\det\begin{pmatrix} a+1 & a & a \\ 2a & 1 & a \\ 2a & a & 1 \end{pmatrix} && \text{Laplace expansion along 1-row} \\
&= (1-a)\det\begin{pmatrix} a+1 & 0 & a \\ 2a & 1-a & a \\ 2a & a-1 & 1 \end{pmatrix} && C_2 \to C_2 - C_3 \\
&= (1-a)\det\begin{pmatrix} a+1 & 0 & a \\ 2a & 1-a & a \\ 4a & 0 & a+1 \end{pmatrix} && R_3 \to R_3 + R_2 \\
&= (1-a)^2\det\begin{pmatrix} a+1 & a \\ 4a & a+1 \end{pmatrix} && \text{Laplace expansion along 2-column} \\
&= (1-a)^2\left((a+1)(a+1) - a\cdot 4a\right) && \text{formula for } 2\times 2\text{-matrieces} \\
&= (1-2a+a^2)(1+2a-3a^2) \\
&= 1-6a^2+8a^3-3a^4.
\end{aligned}
$$

Note that this is not the only possible way to compute the determinant (however the final result should be the same).

Exercise 3 [Theorem 5.18] *Let E be an elementary matrix.*

1. *If E is of type $i)$ then $\det(E) = -1$.*
2. *If E is of type $ii)$ and is obtained from I by multiplying a row (or a column) by a number k, then $\det(E) = k$.*
3. *If E is of type $iii)$, then $\det(E) = 1$.*

Proof. In this proof we will consider elementary matrices obtained from I_n, with $n \in \mathbb{N}$, using a row elementary operation. The case of matrices obtained by elementary column operations can be proved in a symmetric way.

1. See notes.
2. Let us now consider the case of an elementary matrix of type $ii)$. Let i and k, with $1 \leq i \leq n$ and $k \neq 0$ be such that $I \xrightarrow[R_i \to kR_i]{ii)} E$. The matrix

has thus the form

$$E = \begin{array}{cc} \begin{matrix} & & & i & & \end{matrix} & \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} & \begin{matrix} \\ \\ \\ i \\ \\ \\ \end{matrix} \end{array}$$

Using Laplace Expansion Theorem, we can consider the cofactor expansion of E along the i-row and obtain

$$\begin{array}{rcl} \det(E) & = & (-1)^{i+i} \cdot k \cdot \det_{(A_{i,i})} + \sum_{j \neq i} (-1)^{i+j} \cdot 0 \cdot \det(A_{i,j}) \\ & = & k \cdot \det(A_{i,i}) = k \cdot \det(I_{n-1}) \\ & = & k, \end{array}$$

since in $A_{i,i}$ every entrance is 0 except on the main diagonal, where we have 1.

3. Let us finally consider the case of an elementary matrix of type $iii)$. Let i, j and k with $1 \leq i, j \leq n$ and $k \in \mathbb{R}$ be such that $I \xrightarrow[R_j \to R_j + kR_i]{ii)} E$. The matrix is thus of the form

$$E = \begin{array}{cc} \begin{matrix} & & & i & & j & & \end{matrix} & \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} & \begin{matrix} \\ \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} \end{array}$$

Using Theorem 4.32, we can consider the cofactor expansion of E along the i-row and obtain

$$\begin{array}{rcl} \det(E) & = & (-1)^{i+i} \cdot k \cdot \det_{(A_{i,i})} + \sum_{j \neq i} (-1)^{i+j} \cdot 0 \cdot \det(A_{i,j}) \\ & = & 1 \cdot \det(A_{i,i}) = \cdot \det(I_{n-1}) \\ & = & 1, \end{array}$$

since in $A_{i,i}$ every entrance is 0 except on the main diagonal, where we have 1.

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