Linear Algebra with Application (LAWA 2020)

Homework 7

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Exercise 1 [Theorem 5.22, point 2)]

Let $A \in \mathcal{M}_{n,n}(\mathbb{R})$ be an invertible matrix. Then

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}.$$

Proof. Let A^{-1} be the inverse of A. Because of the Product Theorem, we know that

$$\det(A)\det\left(A^{-1}\right) = \det\left(AA^{-1}\right) = \det(I) = 1,$$

Since $\det(A) \neq 0$, from the point 1) of Theorem 5.22, then we can divide both members by $\det(A)$ and obtain

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}.$$

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Exercise 2 [Example 5.23]

Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$ and let us suppose that

$$\det(A) = 2 \quad \text{and} \quad \det(B) = -3.$$

Then we have

$$\begin{aligned}
\det\left(2A^3B^{-1}A^TB^2\right) &= 2^n \det\left(A^3B^{-1}A^TB^2\right) \\
&= 2^n \det\left(A^3\right)\det\left(B^{-1}\right)\det\left(A^T\right)\det\left(B^2\right) \\
&= 2^n \det(A)^3 \tfrac{1}{\det(B)} \det(A)\det(B)^2 \\
&= 2^n \det(A)^4 \det(B) \\
&= -3 \cdot 2^{n+4}.
\end{aligned}$$

Exercise 3 [Example 5.32]

Let us consider the system of linear equations

$$\begin{cases} 3x_1 - x_3 &= 1 \\ 4x_1 + 7x_2 + 3x_3 &= 0 \\ -2x_1 + 8x_2 + 5x_3 &= 1 \end{cases}.$$

We saw in Example 5.29 that the matrix

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 4 & 7 & 3 \\ -2 & 8 & 5 \end{pmatrix}$$

has determinant $\det(A) = -13 \neq 0$ and thus it is invertible. Let us now consider the three matrices

$$A_1(B) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 7 & 3 \\ 1 & 8 & 5 \end{pmatrix}, \quad A_2(B) = \begin{pmatrix} 3 & 1 & -1 \\ 4 & 0 & 3 \\ -2 & 1 & 5 \end{pmatrix} \quad \text{and} \quad A_3(B) = \begin{pmatrix} 3 & 0 & 1 \\ 4 & 7 & 0 \\ -2 & 8 & 1 \end{pmatrix}.$$

Their determinants are respectively

$$\det(A_1(B)) = \det\begin{pmatrix} 7 & 3 \\ 8 & 5 \end{pmatrix} + \det\begin{pmatrix} 0 & -1 \\ 7 & 3 \end{pmatrix} = 11 + 7 = 18,$$

$$\det(A_2(B)) = -\det\begin{pmatrix} 4 & 3 \\ -2 & 5 \end{pmatrix} - \det\begin{pmatrix} 3 & -1 \\ 4 & 3 \end{pmatrix} = -26 - 13 = -39 \quad \text{and}$$

$$\det(A_3(B)) = \det\begin{pmatrix} 4 & 7 \\ -2 & 8 \end{pmatrix} + \det\begin{pmatrix} 3 & 0 \\ 4 & 7 \end{pmatrix} = 46 + 21 = 67.$$

Thus, using Cramer's Rule, we have

$$x_1 = -\frac{18}{13}, \quad x_2 = \frac{39}{13} = 3 \quad \text{and} \quad x_3 = -\frac{67}{13}.$$