Chapter 1

MPI - lecture 7

Content of lecture

- Elliptic curves;
- Fields, rings;
- Finite fields in general;
- Binary fields.

1.1 Elliptic curves

Calculation on elliptic curves

On elliptic curves, we use points (x, y) with x and y being the residue classes modulo some prime number p.

The operation, which is usually denoted by $+$, is defined as follows:

Definition 1. *For two points $\mathcal{P} = (x_1, y_1)$ and $\mathcal{Q} = (x_2, y_2)$, we define $\mathcal{P} + \mathcal{Q} = (x_3, y_3)$ as:*

$$\begin{aligned} x_3 &\equiv s^2 - x_1 - x_2 \ (mod\ p) \\ y_3 &\equiv s(x_1 - x_3) - y_1 \ (mod\ p) \end{aligned}$$

where

$$s = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} \ (mod\ p) & \text{if } \mathcal{P} \neq \mathcal{Q} \\ \dfrac{3x_1^2 + a}{2y_1} \ (mod\ p) & \text{if } \mathcal{P} = \mathcal{Q}. \end{cases}$$

The parameter a is taken from the equation of a given elliptic curve $y^2 \equiv x^3 + ax + b \ (mod\ p)$, with $a, b \in \mathbb{Z}_p$, which must by fulfilled by all points. The neutral element $\mathcal{O}$ is "artificially" defined such that it has the properties of the neutral element.

Elliptic curves over the field of real numbers (1/3)

To better understand what is happening on elliptic curves, we consider points (x, y) from the *continuous* plane $\mathbb{R}^2$.

Definition 2. *An elliptic curve is the set of points given by the equation*

$$y^2 = x^3 + ax + b,$$

where the real coefficients a, b satisfy $4a^3 + 27b^2 \neq 0$.

The group operation can be defined by geometrical means (see Wolfram Demonstrations Project).

Elliptic curves over the field of real numbers (2/3)

When we add two poinsts $\mathcal{P} = (x_1, y_1)$ and $\mathcal{Q} = (x_2, y_2)$, we plot a line through $\mathcal{P}$ and $\mathcal{Q}$ and we look for intersections of this line and the elliptic curve.

If $\mathcal{P} \neq \mathcal{Q}$, the line going through $\mathcal{P}$ and $\mathcal{Q}$ has equation

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2) + y_2.$$

If $\mathcal{P} = \mathcal{Q} = (x_1, y_1)$, a tangent line to the elliptic curve at $\mathcal{P} = \mathcal{Q}$ has slope given by the derivative of the curve

$$y = \sqrt{x^3 + ax + b}$$

at x_1, which is

$$\frac{3x^2 + a}{2\sqrt{x^3 + ax + b}} = \{\text{for } x = x_1\} = \frac{3x_1^2 + a}{2y_1},$$

and thus the equation of the line is

$$y = \frac{3x_1^2 + a}{2y_1}(x - x_1) + y_1\,.$$

Elliptic curves over the field of real numbers (3/3)

Now, we are looking for intersections of some line $y = sx + d$, where s is the slope and $d \in \mathbb{R}$, and the elliptic curve $y^2 = x^3 + ax + b$.

This leads to solving the equation

$$(sx + d)^2 = x^3 + ax + b\,,$$

i.e., a polynomial equation of order 3, witch can have 1, 2 or 3 real roots (we have always at least one solution; namely the points $\mathcal{P}$ and $\mathcal{Q}$).

For instance, the situation where we obtain only two roots for distinct $\mathcal{P}$ and $\mathcal{Q}$ corresponds to $\mathcal{Q} = -\mathcal{P}$ and the result of the sum is the neutral element $\mathcal{O}$.

In general, the solution of the equation is obtained in this way: For two points $\mathcal{P} = (x_1, y_1)$ and $\mathcal{Q} = (x_2, y_2)$ define $\mathcal{P} + \mathcal{Q} = (x_3, y_3)$ as:

$$x_3 = s^2 - x_1 - x_2 \quad \text{and} \quad y_3 = s(x_1 - x_3) - y_1$$

where

$$s = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} & \text{if } \mathcal{P} \neq \mathcal{Q} \\ \dfrac{3x_1^2 + a}{2y_1} & \text{if } \mathcal{P} = \mathcal{Q}\,. \end{cases}$$

Elliptic curves over a general field

The previous three slides show how to define a group over the set of points in the plane $\mathbb{R} \times \mathbb{R}$.

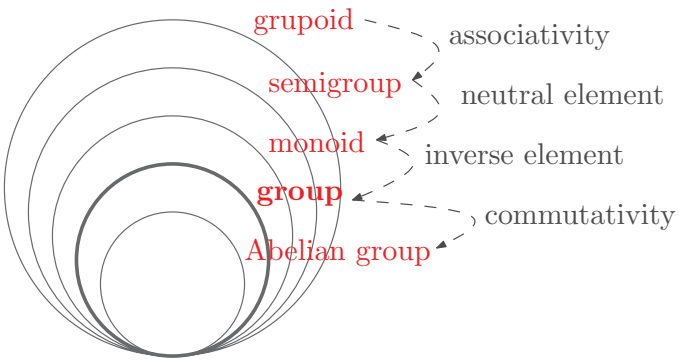
Generally, instead of the set $\mathbb{R}$ we could take an arbitrary field, e.g., the field $\mathbb{Z}_p$, where the discrete logarithm problem is difficult to solve.

Field, what is that? Generally speaking, it is a set with two binary operations (usually called *addition* and *multiplication*) which allow us to define analogues of common arithmetical operations such as subtracting, dividing, rising to the power, logarithm,...

1.2 Rings and fields

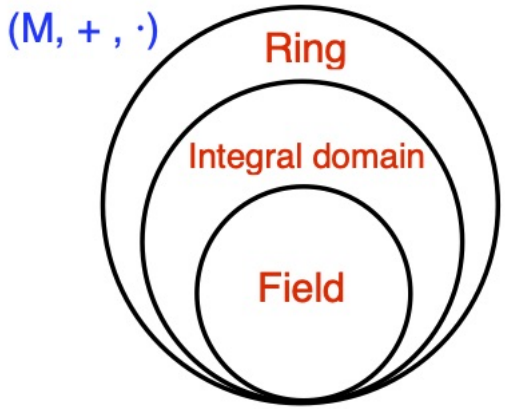
Sets with one binary operation

A nonempty set M with a binary operation $\cdot$ (resp. $+$ for the additive notation).



Sets with two binary operations

For more sophisticated arithmetical operations with numbers we need **both** addition and multiplication.



Definition of a ring

Definition 3 (Ring)**.** *Let M be a nonempty set, and $+$ and $\cdot$ two binary operations. We say that $R = (M, +, \cdot)$ is a ring if the following holds:*

- $(M, +)$ *is an* ***Abelian group****,*
- $(M, \cdot)$ *is a* ***monoid****,*

- *both left and right* ***distributive law*** *hold:*

$$(\forall\, a, b, c \in M) \text{ we have: } \quad a(b+c) = ab + ac \quad \wedge \quad (b+c)a = ba + ca.$$

We respect the standard convention that the multiplication has a higher priority than the addition.

Sometimes, $(M, \cdot)$ is required to be only a semigroup.

Terminology

Let $R = (M, +, \cdot)$ be a ring.

- If $\cdot$ is associative, R is an associative ring.
- If $\cdot$ is commutative, R is a commutative ring.
- $(M, +)$ is called the additive group of the ring R.
- $(M, \cdot)$ is called the multiplicative groupoid/monoid of the ring R.
- The neutral element of the group $(M, +)$ is called the zero element and is denoted by 0; the inverse element to $a \in M$ is denoted as $-a$.
- Inside the ring **we can define subtraction** by

$$a - b := a + (-b).$$

Examples

- $(\mathbb{N}, +, \cdot)$ is not a ring, because $(\mathbb{N}, +)$ is not a group.
- $(\mathbb{Z}, +, \cdot)$ is a ring.
- The trivial ring is $(\{0\}, +, \cdot)$ (if it holds that $0 \cdot 0 = 0$).
- The set $(\mathbb{R}^{n,n}, +, \cdot)$ of square real matrices with the usual addition and multiplication is a ring; the zero element is the zero matrix.
- The set of all polynomials (with complex / real / integer coefficients) is a ring; the zero element is the zero polynomial $p(x) = 0$.

Basic properties of rings

In an arbitrary ring $(M, +, \cdot)$, the following holds.

- Left and right distributive law for subtracting, i.e.,

$$c(b-a) = cb - ca.$$

Indeed:

$$ca + c(b-a) = c(a+b-a) = cb \quad \Longrightarrow \quad c(b-a) = cb - ca.$$

□

- Multiplying by the zero element returns the zero element, i.e.,

$$\forall a \in M \qquad a \cdot 0 = 0 \wedge 0 \cdot a = 0.$$

Indeed:

$$a \cdot 0 = a(a-a) = aa - aa = 0.$$

□

Integral domain

Definition 4 (zero divisors)**.** *Let $R = (M, +, \cdot)$ be a ring. Two arbitrary **nonzero** elements $a, b \in M$ such that*

$$a \cdot b = 0$$

are called zero divisors.

Definition 5 (integral domain)**.** *A commutative ring **without** zero divisors is called an integral domain.*

Examples of integral domains

- $(\mathbb{Z}, +, \cdot)$ is an integral domain.

- Each number ring $(M, +, \cdot)$, where $M \subset \mathbb{C}$ and $+$ and $\cdot$ are classical, is an integral domain.
- The ring $(\mathbb{R}^{n,n}, +, \cdot)$ is **not** an integral domain for $n \geq 2$, because it is not commutative; moreover, it has zero divisors:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition of field

Definition 6 (field)**.** *A ring $T = (M, +, \cdot)$ is a field if $(M \setminus \{0\}, \cdot)$ is an Abelian group. This group is called the multiplicative group of the field T.*

Why do we have to remove the zero element?

Because the zero has no inverse element (with respect to the multiplication), i.e., it is not possible to divide by zero: $0^{-1} = ?!$.

We can divide by all other elements of the field!

dividing = multiplying by the inverse element

$$\frac{a}{b} := a \cdot b^{-1} \quad \text{for } b \neq 0.$$

Examples of fields

- The ring of integers $(\mathbb{Z}, +, \cdot)$ is **not** a field, because $(\mathbb{Z} \setminus \{0\}, \cdot)$ misses some inverse elements.
- The ring of rational numbers $(\mathbb{Q}, +, \cdot)$ is a field. Moreover, it is the smallest number field (with the common arithmetical operations).
- The smallest field is the so-called trivial field $(\{0, 1\}, +, \cdot)$, with operations given by the following tables:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

The first table corresponds to the bit operation `XOR` and the latter to `AND`, or, alternatively, to the addition and multiplication modulo 2.

Some properties

In each field all usual arithmetical operations are defined: *addition*, *subtraction*, *multiplication*, *division*, and all operations derived from them such as *rising to the power*, *root extractions*, *logaritm*, ...

Using the trivial field we have all these operations over one bit. Later we will show how to extend them to any number of bits.

Theorem 7. *Each field is an integral domain.*

Proof. Since the multiplicative group of the field $(M \setminus \{0\}, \cdot)$ is closed under multiplication, for all nonzero a, b it holds that their product $a \cdot b \in M \setminus \{0\}$ is again nonzero. □

Homomorphism and isomorphism

Definition 8. *A mapping h from the ring (resp. field) R_1 to the ring (resp. field) R_2 is a homomorphism if h is a homomorphism of the corresponding additive and multiplicative groupoids (resp. groups).*

If, moreover, h is bijective (injective and surjective), it is an isomorphism.

1.3 Finite fields

Finite fields

A field with finite number of elements is called finite. The number of elements is said to be the order of the field.

An example of finite field is the set (of residue classes modulo p)

$$\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$$

with operations modulo a **prime** p (see the previous lecture).

E.g., for $p = 5$ we obtain the field with following operations:

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

and

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Additive group $(\mathbb{Z}_p, +)$

- The order of the additive group $(\mathbb{Z}_p, +)$ is the prime number p.
- Each nonzero element is a generator (this holds for all groups with prime order).
- $(\mathbb{Z}_p, +)$ is a group even when p is not prime.

Multiplicative group $(\mathbb{Z}_p \setminus \{0\}, \cdot)$

- The order of the group $\mathbb{Z}_p^\times$ is $p-1$ and this is never a prime number for $p \neq 3$!
- $\mathbb{Z}_p^\times$ is cyclic (i.e., there exists a generator of it).
- The number of generators depends on $p-1$, and is equal to the number of numbers coprime to $p-1$, i.e., $\varphi(p-1)$.
- If k with $k < p$ divides $p-1$, then there exists a subgroup in $\mathbb{Z}_p^\times$ of order k and it contains just the elements for which $a^k = 1$.

Orders of fields?

We have shown a construction of a finite field of order p with p prime.
Are there fields of any arbitrary order?

Theorem 9. *Any finite field has* ***order*** $\mathbf{p^n}$*, where p is a prime number and n is a positive natural number.*
The prime number p is called the characteristic of the field.
Furthermore, all fields of order p^n are isomorphic.
Additionally, the multiplicative group of a finite field is cyclic.

Consequence: There are no fields of order 6, 10, 12, 14, ...

If we chose $p = 2$ and $n = 8$, we obtain the field providing us with arithmetic on 1 byte (8 bits)!

1.4 Binary fields

Symmetric cryptography (more the course MIE-BHW)

Secure exchange of longer text performed by asymmetric ciphers (RSA, Diffie-Hellman and others) is not effective.

That is why the **symmetric ciphers** are used: symmetric ciphers assume that (and take advantage of) Alice and Bob share some secret key.
Asymmetric ciphers are used only for exchanging this private key.

A very common method is the block cipher called Advanced Encryption Standard (AES).
Here we get acquainted with the mathematics underlying this method.

AES block cipher

The text we want to securely transfer is divided into (e.g.) blocks having 8 bits. Then, these blocks are encoded using the shared key so that the decoding can be easily made using the same key.

This cipher AES is based on the fact that arithmetic operations with $n = 8$ bits can be understood as operations in a finite field with 2^n elements for $n = 8$.
The fields with 2^n elements are called binary fields and are denoted $GF(2^n)$ (as *Galois Fields*).

We now explain how to define addition and multiplication in these fields.

The wrong way

Consider a field $GF(2^8)$. Each element can be represented as an 8 bit string, e.g., 11010110, 01100011, etc.

Addition: Addition can be defined component-wise modulo 2. i.e.

$$11010110 + 01100011 = (1+0 \mod 2)(1+1 \mod 2)\cdots \\ \cdots(0+1 \mod 2) = 10110101.$$

The neutral (zero) element is 00000000, and each element is inverse to itself. We have an additive group.

Multiplication: Multiplication cannot be defined component-wise: The neutral element would be 11111111 and the inverse to (e.g.) 11111110 would not exist.

Multiplication must be defined in a different way!

Rings of polynomials over a ring / field

In order to be able to add, subtract, and multiply a polynomial of the form $\sum a_i x^i$, we only need to know how to add, subtract, and multiply the coefficients. In general, we can construct a ring of polynomials over an arbitrary ring or field similar to the one we know from real or complex numbers.

Definition 10. *Let K be a ring. The set of polynomials with coefficients in K together with operations of addition and multiplication defined as*

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i;$$

$$\left(\sum_{i=0}^{n} a_i x^i\right) \cdot \left(\sum_{i=0}^{m} b_i x^i\right) = \sum_{i=0}^{n+m} \left(\sum_{j+k=i} a_j b_k\right) x^i$$

is the commutative ring of polynomials over the ring K. This ring is denoted as $K[x]$.

Irreducible polynomial

Definition 11. *Let K be a field and $P(x) \in K[x]$ be of degree at least 1. We say that $P(x)$ is irreducible over K if, for any two polynomials $A(x)$ and $B(x)$*

from $K[x]$*, it holds that*

$$A(x) \cdot B(x) = P(x) \quad \Rightarrow \quad (\text{degree of } A(x) = 0 \quad \vee \quad \text{degree of } B(x) = 0).$$

Irreducible polynomials are primes among polynomials!
Their definition is analogous as well as their properties.

Example: Whereas $x^2 + 1$ is irreducible over the field $\mathbb{Q}$, the polynpmial $x^2 - 1 = (x + 1)(x - 1)$ is not.

Remark: $x^2 + 1$ is irreducible over the field $\mathbb{Q}$, but not over the field $\mathbb{Z}_2$, where the coefficients are added and multiplied modulo 2:

$$x^2 + 1 = (x + 1)(x + 1) = x^2 + 2x + 1.$$

Irreducible polynomial as a modulus

We define modulo polynomial as:

$$A(x) \pmod{P(x)} = \text{the remainder of the division of } A(x) \text{ by } P(x).$$

The result is always a polynomial of degree less than the degree of $P(x)$.

Example: for $A(x) = x^3$ and $P(x) = x^2 + 1$ we have $A(x) = x(x^2 + 1) + (-x)$ and thus

$$x^3 \equiv -x \pmod{x^2 + 1}.$$

If $P(x)$ is irreducible (with respect to the field from which the coefficients are taken), the remainders after division by $P(x)$ form a group (if we again remove the zero polynomial).

Field $GF(2^4)$

The elements $GF(2^4)$ are represented as polynomials of order at most 3 with coefficients h_i from the field $\mathbb{Z}_2$:

$$h_3x^3 + h_2x^2 + h_1x + h_0 \approx (h_3h_2h_1h_0)_2.$$

Addition component-wise modulo 2:

$$(x^3 + x + 1) + (x^2 + x + 1) = x^3 + x^2.$$

Field $GF(2^4)$ – multiplication

Multiplication modulo a chosen irreducible polynomial, e.g., $x^4 + x + 1$.
Example: multiplication $A(x) \cdot B(x)$ for $A(x) = x^3 + x^2 + 1$ and $B(x) = x^2 + x$.

1. Multiply $A(x) \cdot B(x)$ classically and rewrite coefficients $\mod 2$:

$$A(x) \cdot B(x) = x^5 + 2x^4 + x^3 + x^2 + x = x^5 + x^3 + x^2 + x.$$

2. Find the remainder after division by $P(x)$. Since

$$x^5 = x(x^4 + x + 1) + (x^2 + x), \quad \text{it holds} \quad x^5 \equiv x^2 + x \pmod{x^4 + x + 1},$$

and we have

$$x^5 + x^3 + x^2 + x \equiv (x^2 + x) + (x^3 + x^2 + x) \equiv \mathbf{x^3} \pmod{x^4 + x + 1}.$$

Hence we get that $1101 \cdot 0110 = 1000$.

AES in field $GF(2^8)$

According to the specification of AES, the multiplication is done modulo

$$x^8 + x^4 + x^3 + x + 1.$$

1.5 Construction of a general finite field

Construction of a finite field

In general, we construct a finite field $GF(p^k)$ using polynomials as follows.
Let $m(x) \in \mathbb{Z}_p[x]$ be an irreducible polynomial of degree k.

$$GF(p^k) = \left(\{q(x) \in \mathbb{Z}_p[x] \colon \deg(q) < k\}, \ +, \ \times \bmod m(x) \right).$$