# Chapter 1

# MPI - lecture 9

## 1.1 Eigenvectors

Eigenvalues and eigenvectors

A complex number $\lambda$ is called an eigenvalue of the matric $M \in \mathbb{C}^{n,n}$, whenever there exists a non-zero vector $u \in \mathbb{C}^n$ such that

$$Mu = \lambda u.$$

The vector $u$ is called an eigenvector of the matrix $M$ relative to the eigenvalue $\lambda$.

The set of eigenvectors of $M$ (relative to the eigenvalues $\lambda$ and to the zero vector) form a base of the subspace $\ker(M - \lambda E)$.

The eigenvalues of the matrix $M$ are the roots of the characteristic polynomial of the $M$, that is the polynomial

$$p_M(\lambda) := \det(M - \lambda E).$$

Therefore, each matrix $M \in \mathbb{C}^{n,n}$ has at most $n$ different complex eigenvalues.

Diagonalizability of a matrix

A matrix $M \in \mathbb{C}^{n,n}$ is diagonalizable when there exist a diagonal matrix $D \in \mathbb{C}^{n,n}$ and a regular matrix $P \in \mathbb{C}^{n,n}$ such that

$$M = PDP^{-1}.$$

where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

**Remind**: In the previous lecture we saw that $M^k = PD^kP^{-1}$.

**Remark**: The columns of the matrix $P$ are the eigenvectors of $M$. These eigenvectors form a basis of $\mathbb{C}^n$. The elements of the diagonal matrix $D$ are the eigenvalues of $M$ (with their multiplicity).

## Looking for an eigenvector

Let $M \in \mathbb{C}^{n,n}$. Suppose it is diagonalizable and we can order its eigenvalues as follows

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|.$$

We are looking for the eigenvector of the eigenvalue $\lambda_1$, the so-called dominant eigenvalue. It is a vector $u_1$ such that

$$Mu_1 = \lambda_1 u_1.$$

In general, the matrix need not be diagonalizable, but the ideas would be more complicated (actually, we only require to have one eigenvalue which is the greatest in absolute value).

## Applications

Eigenvalues play an importan role in several applications:

- Classification of conics and quadratic forms (geometry).
- Quantum computation, quantum mechanics, asymptotic behaviour of dynamical systems (physics).
- PCA, or *Principal Component Analysis* (big data).
- Recognition of 2D and 3D objects using spectral methods (AI).
- More practical example: **PageRank** mesures a relative importance of WWW documents by ispecting links between them.
  - Its values is in fact an eigenvector of the dominant eigenvalues of a modified adjacency matrix of these links. This matrix satisfies requirement of our problem.
  - **PageRank** is calculated using power methods.

## 1.2 Power method

Introduction and assumptions (1/2)

In its basic variant, the power method is used to find the dominant eigenvalue of a matrix,

Given a matrix $M \in \mathbb{C}^{n,n}$ let us consider a regular matrix $P \in \mathbb{C}^{n,n}$ such that

$$M = PDP^{-1}$$

where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Let also suppose that the values are ordered:

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

**Note**: We suppose that the dominant eigenvalue $\lambda_1$ is not degerate (i.e., that the correspoinding eigenspace has dimension 1).

Introduction and assumptions (2/2)

We are looking for an eigenvector associated to the eigenvalue $\lambda_1$, that is a non-zero vector $u_1$ such that

$$Mu_1 = \lambda_1 u_1.$$

The power method is an **iterative method**. We will construct a sequence $(x_k)$ as follows: $x_0$ is chosen randomly and the next terms are determined by

$$x_k = Mx_{k-1} \quad \text{for } k > 0.$$

Equivalently, we have

$$x_k = M^k x_0 \quad k \in \mathbb{N}_0.$$

Power method principle (1/4)

If $M$ is regular, thus diagonalizable, there exist eigenvectors $\{u_1, u_2, \ldots, u_n\}$, which form a basis of $\mathbb{C}^{n,1}$.

If $M$ is not regular, then we need to complete the set of eigenvectors by a basis of the kernel of $M$.

The vector $x_0$ can be written as $x_0 = \alpha_1 u_1 + \cdots + \alpha_n u_n$.

Suppose that $\alpha_1 \neq 0$.

Coefficients $\alpha_i$ can be absorbed by the eigenvectors ($u_i' = \alpha_i u_i$) and we have

$$x_0 = u_1' + \cdots + u_n'.$$

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Power method principle (2/4)

The recurrent definition of $x_k$ implies

$$\begin{aligned} x_k &= M^k x_0 \\ &= M^k u_1 + \cdots + M^k u_n \\ &= \lambda_1^k u_1 + \cdots + \lambda_n^k u_n. \end{aligned}$$

The last equality gives

$$x_k = \lambda_1^k \left( u_1 + \left( \frac{\lambda_2}{\lambda_1} \right)^k u_2 + \cdots + \left( \frac{\lambda_n}{\lambda_1} \right)^k u_n \right).$$

We rewrite it as

$$x_k = \lambda_1^k \left( u_1 + \varepsilon_k \right).$$

Since for all $j > 1$ we have $\left| \frac{\lambda_j}{\lambda_1} \right| < 1$, then $\lim_{k \to +\infty} \varepsilon_k = 0$.

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Power method principle (3/4)

The sequence $\left( \frac{x_k}{\lambda_1^k} \right)$ "converges" to the eigenvector of the dominant eigenvalues.

We have $\|x_k\| \to +\infty$. Thus we need to control the norm: we may set it to 1 at each step (by *normalizing*, i.e., considering $y_k = \frac{x_k}{\|x_k\|}$).

To have convergence also for the case $\lambda_1 < 0$, we need to pick the right direction for the eigenvector so that it does not oscillate. We may do this by setting the largest entry in absolute value to 1 (and thus use the maximum norm).

The speed of convergence is given by $\lambda_2$ since $\|\varepsilon_k\| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$

Power method principe (4/4)

How to find the dominant eigenvalue?

If $\varphi$ is a linear mapping $\varphi : \mathbb{C}^{n,1} \mapsto \mathbb{C}$ such that $\varphi(u_1) \neq 0$, then

$$\frac{\varphi(x_{k+1})}{\varphi(x_k)} = \frac{\varphi\left(\lambda_1^{k+1}\left(u_1 + \varepsilon_{k+1}\right)\right)}{\varphi\left(\lambda_1^k\left(u_1 + \varepsilon_k\right)\right)} = \frac{\lambda_1^{k+1}\left(\varphi(u_1) + \varphi(\varepsilon_{k+1})\right)}{\lambda_1^k\left(\varphi(u_1) + \varphi(\varepsilon_k)\right)} \quad \to \quad \lambda_1 \quad \text{for } k \to +\infty.$$

The mapping $\varphi$ can be set to the mapping defined for all $x \in \mathbb{C}^{n,1}$ as $\varphi(x) = x_{(1)}$ where $x_{(1)}$ is the first component $x$ (if $\varphi(u_1) \neq 0$)).

Power method - demonstration in $\mathbb{R}^{n,n}$

Let us find the dominant eigenvector of $M = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$, which satisfies the conditions of power method.

The exact solution is $u_1 = (1, \sqrt{2}+1) = \frac{1}{\sqrt{2}+1}(\sqrt{2}-1, 1)$, with eigenvalue $\lambda_1 = 3 + \sqrt{2}$.

| $k$ | $\widehat{x}_k$ | $\|\widehat{x}_k - \widehat{x}_{k-1}\|_\infty$ |
|---|---|---|
| 0 | $(1.0, 1.0)$ | - |
| 1 | $(0.59999999999999998, 1.0)$ | 0.4 |
| 2 | $(0.47826086956521746, 1.0)$ | 0.121739130435 |
| 3 | $(0.43689320388349517, 1.0)$ | 0.0413676656817 |
| 4 | $(0.42231947483588622, 1.0)$ | 0.0145737290476 |
| 5 | $(0.4171202375061851, 1.0)$ | 0.0051992373297 |

In the calculations, the maximum entry in absolute value is set to 1 at each step and the convergence criterion $\|\widehat{x}_k - \widehat{x}_{k-1}\|_\infty < 10^{-2}$.

Power method - demonstration in $\mathbb{C}^{n,n}$ (1/2)

Let us consider the matrix

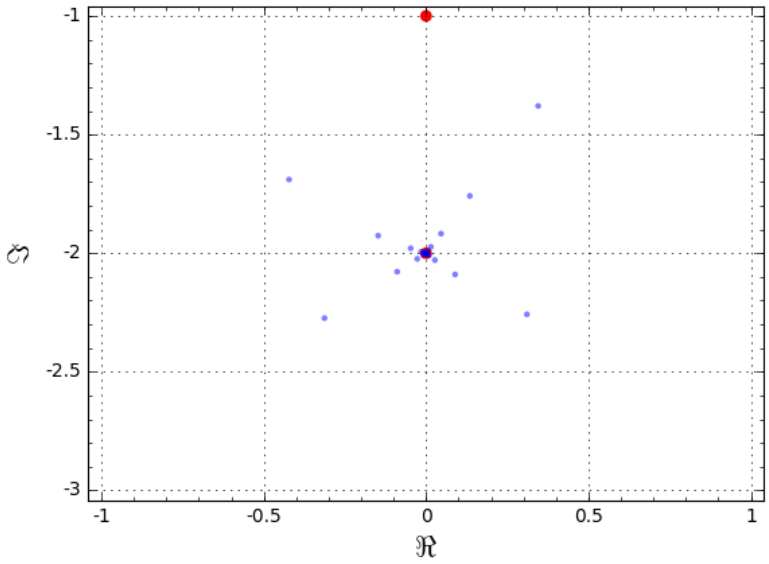
$$M = \begin{pmatrix} 36408 + 16769i & -5412 - 2481i & 107256 + 49397i & -492 - 214i \\ -10656 - 5164i & 1584 + 762i & -31392 - 15210i & 144 + 66i \\ -12876 - 5954i & 1914 + 881i & -37932 - 17539i & 174 + 76i \\ 4329 - 262i & -643 + 39i & 12753 - 771i & -58 + 6i \end{pmatrix}$$

The eigenvalues are $-2i$, $-i$, $3i/2$ and $3/2$.

Let us fix the accuracy at $\varepsilon = 10^{-6}$. The last 7 iterations of $\lambda_1^{(k)}$ are:

0.0000477588150960872 - 1.99991424541241 $i$
-0.0000479821875446196 - 1.99998019901599 $i$
-0.0000272650944159076 - 2.00002375338328 $i$
0.0000271520045767515 - 2.00002973125038 $i$
0.0000154506695115737 - 1.99997272532314 $i$
-0.0000152424622193764 - 1.99999349337182 $i$

Power method - demonstration in $\mathbb{C}^{n,n}$ (2/2)




Power method: other eigenvalues

Suppose that using the power method we found the dominant eigenvalue $\lambda_1$ and its correspoding (normalized) eigenvector $u_1$. How can we find the other eigenvalues?

Suppose that the matrix $M$ is normal (i.e., that $MM^* = M^*M$, where $M^*$ is the conjugate transpose of $M$). Then its eigenvectors are orthogonal.

We can consider a new matrix $M'$ defined as:

$$M' := M - \lambda_1 u_1 \cdot u_1^*$$

The matrix $M'$ has $u_1$ as eigenvector, but the associated eigenvalue is $0$, indeed:

$$M'u_1 = Mu_1 - \lambda_1 u_1 \cdot \|u_1\|^2 = \lambda_1 u_1 - \lambda_1 u_1 = 0.$$

We can now apply the power method to the matrix $M'$. The dominant eigenvalue of $M'$ will be the second largest (in absolute value) eigenvalue of $M$.

## 1.3 QR algoritmus

QR factorization and QR algorithm (1/2)

The power method is not suitable to find all eigenvalues of a given matrix $M$.

Other algorithms are based on the fact that similar matrices have the same eigenvalues. The goal of QR algorithm is to construct a sequence $(M_k)_{k=0}^{\infty}$ of similar matrices in the following way:

$$M_0 = M \quad \text{and} \quad M_k = P_k M_{k-1} P_k^{-1} \; k \in \mathbb{N},$$

where each $P_k$ is a regular matrix, $M_k \to M_\infty$ and for $M_\infty$ is easy to find the eigenvalues (for instance, $M_\infty$ is upper triangular).

QR factorization and QR algorithm (2/2)

The QR factorization consists in expressing a real (or complex) matrix $M \in \mathbb{R}^{n,n}$ as a product

$$M = Q \cdot R$$

where $Q$ is an orthogonal matrix (unitary if $M \in \mathbb{C}^{n,n}$) and $R$ is upper triangular.

There exist several algorithms to compute such a factorization (Gram-Schmidt, LR algorithm, ...)

The QR algorithm consists in applying such a factorization at any step, that is for every $k$ we have

$$M_k = Q_k \cdot R_k$$

and we define

$$M_{k+1} := R_k Q_k = Q_k^{-1} Q_k R_k Q_k = Q_k^{-1} M_k Q_k.$$

We start the iteration with $M_0 = M$. Every matrix $M_k$ is similar to the previous matrix $M_{k-1}$ in the sequence, so that all matrices have the same eigenvalues.

Under certain conditions $M_k$ converges to a triangular matrix.