# Mathematics for Informatics

## Constrained Optimization, Multivariate Integration (lecture 3 of 12)

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Fall 2019/2020

created: December 9, 2019, 18:22

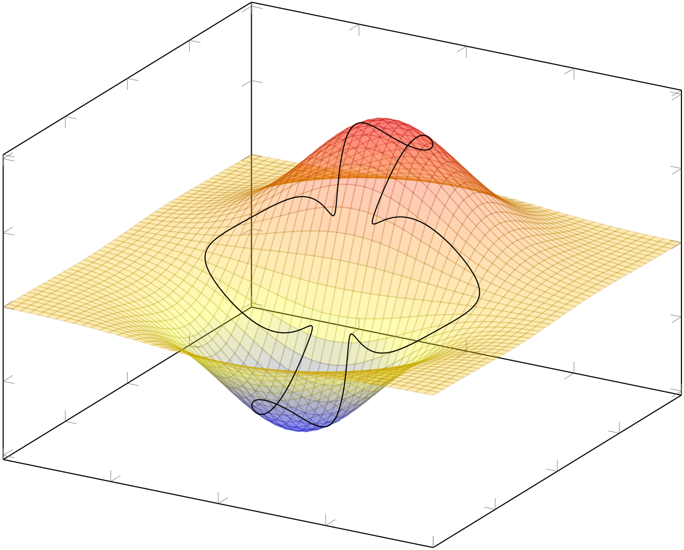
# Outline

- Constrained optimization
- Reminder: integration of functions of 1 variable
- 2-variate function integration

# Constrained Optimization, Multivariate Integration

# Motivation

Find the maximum and minimum points when walking along the black line:



# The problem

Let $f : \mathbb{R}^n \to \mathbb{R}$. Find (local) maxima and minima of $f$ subject to

$$\begin{cases} g_1(x_1, x_2, \dots, x_n) & = & 0 \\ g_2(x_1, x_2, \dots, x_n) & = & 0 \\ & \vdots & \\ g_p(x_1, x_2, \dots, x_n) & = & 0. \end{cases}$$

Set

$$\mathcal{G} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid g_i(x_1, x_2, \dots, x_n) = 0, i = 1, 2, \dots, p\}.$$

# Assumptions

1. The functions $f$ and $g_i$, with $i = 1, 2, \ldots, p$, have continuous second partial derivatives.

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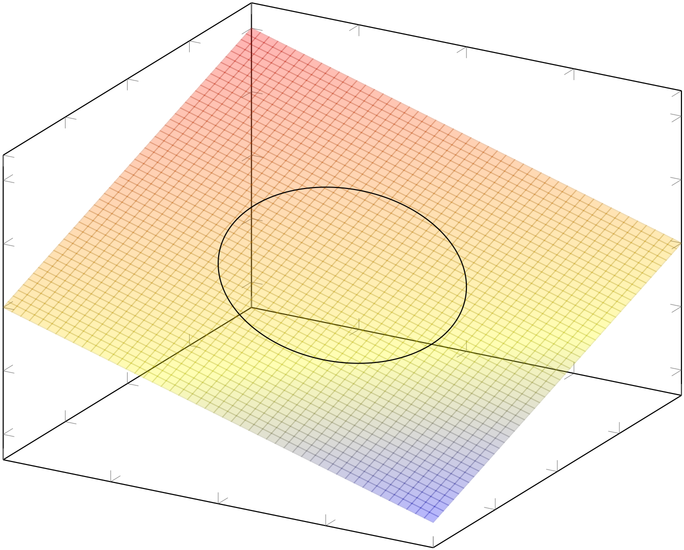
## Example

*Are the gradients of the following functions linearly independent?*

$$g_1(x,y) = 2x + xy^2, \quad g_2(x,y) = 4x + 2xy^2,$$
$$g_3(x,y) = 2xy^2 + 4y^2, \quad g_4(x,y) = 2x + 3xy^2 + 4y^2.$$

# Running example

$$f(x, y) = x + y, \qquad \mathcal{G} = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 2\}$$



# Necessary condition

## Theorem

*Assume $f$ has a local extremum in $x^* = (x_1^*, \ldots, x_n^*) \in \mathcal{G}$ subject to $\mathcal{G}$.*
*Then there exist numbers $\mu_1^*, \ldots, \mu_p^*$ such that the Lagrangian function $L$ given by*

$$L(x_1, \ldots, x_n, \mu_1, \ldots, \mu_p) = f(x_1, \ldots, x_n) + \sum_{i=1}^{p} \mu_i g_i(x_1, \ldots, x_n)$$

*has zero partial derivatives with respect to $x_1, \ldots, x_n$ at the point $x^*$.*
*In other words, the following system of equations is true:*

$$\begin{cases} \dfrac{\partial f}{\partial x_1}(x^*) + \mu_1^* \dfrac{\partial g_1}{\partial x_1}(x^*) + \cdots + \mu_p^* \dfrac{\partial g_p}{\partial x_1}(x^*) & = & 0 \\ & \vdots & \\ \dfrac{\partial f}{\partial x_n}(x^*) + \mu_1^* \dfrac{\partial g_1}{\partial x_n}(x^*) + \cdots + \mu_p^* \dfrac{\partial g_p}{\partial x_n}(x^*) & = & 0 \end{cases}$$

# Sufficient condition

## Theorem

*Let* $x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ *and* $\mu^* = (\mu_1^*, \ldots, \mu_p^*) \in \mathbb{R}^p$ *such that*

- (i) *the Lagrangian function* $L(x_1, \ldots, x_n, \mu_1, \ldots, \mu_p)$ *has zero partial derivatives with respect to* $x_1, \ldots, x_n$ *at the point* $(x^*, \mu^*) \in \mathbb{R}^{n+p}$;
- (ii) *the Lagrangian function* $L(x_1, \ldots, x_n, \mu_1, \ldots, \mu_p)$ *has zero partial derivatives with respect to* $\mu_1, \ldots, \mu_p$ *at the point* $(x^*, \mu^*) \in \mathbb{R}^{n+p}$;
- (iii) *for all non-zero* $y \in \mathbb{R}^n$ *satisfying* $y \cdot \nabla g_i(x^*) = 0$ *for* $i = 1, 2, \ldots, p$ *we have*

$$y\left(\nabla^2 f(x^*) + \sum_{i=1}^{p} \mu_i^* \nabla^2 g_i(x^*)\right) y^T > 0.$$

*Thus, the function* $f$ *has a strict local minimum at* $x^*$ *(subject to* $\mathcal{G}$*).*

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2. (ii) *the Lagrangian function* $L(x_1, \dots, x_n, \mu_1, \dots, \mu_p)$ *has zero partial derivatives with respect to* $\mu_1, \dots, \mu_p$ *at the point* $(x^*, \mu^*) \in \mathbb{R}^{n+p}$*;*
3. (iii) *for all non-zero* $y \in \mathbb{R}^n$ *satisfying* $y \cdot \nabla g_i(x^*) = 0$ *for* $i = 1, 2, \dots, p$ *we have*

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*Thus, the function* $f$ *has a strict local minimum at* $x^*$ *(subject to* $\mathcal{G}$*).*

If we replace in (iii) the condition "$> 0$" by "$< 0$", we obtain a sufficient condition of a strict local maximum.

# Example

**Example**

*Find maxima and minima of* $f(x, y) = x + y$ *subject to* $x^2 + y^2 = 2$.

# Constrained Optimization, Multivariate Integration

# Integration of functions of 1 variable

Let $f : \mathbb{R} \to \mathbb{R}$ and $a < b$.

Recall what does $\int_a^b f(x)\,\mathrm{d}x$ mean, if it exists.

What is its geometrical meaning?

# Definition

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The upper Darboux sum of $f$ with respect to the partition $\Delta$ is

$$S_{f,\Delta} = \sum_{i=1}^{n} F_{\Delta,i}(x_i - x_{i-1})$$

and the lower Darboux sum of $f$ with respect to the partition $\Delta$ is

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# Definition...

The upper Darboux integral (of $f$ over $[a, b]$) is

$$D_f = \inf\{S_{f,\Delta} \colon \Delta \text{ is a partition of } [a, b]\}$$

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If $D_f = d_f$, we call this value the Darboux integral of $f$ over $[a,b]$, and denote it

$$\int_a^b f(x)\,\mathrm{d}x = D_f = d_f.$$

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We say that $f$ is (Darboux-)integrable over $[a, b]$.

This is equivalent to the Riemann integral and to Riemann integrability.

# A few properties

If $f$ is continuous on $[a, b]$, then it is integrable on $[a, b]$.

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Let $f$ be integrable on $[a, b]$ and on $[b, c]$ (with $a < b < c$).
We have that $f$ is integrable on $[a, c]$ and

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x.$$

# Primitive function

Let $F(x)$ be a real function which is continuous on $[a, b]$ and differentiable on $(a, b)$
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$$\forall x \in (a, b), \quad F'(x) = f(x).$$

Such function $F$ is called a primitive function of $f$ on $(a, b)$.

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**Example**

*Find a primitive function on* $(0, 1)$ *of the function* $f(x) = 2x + x^2$.

# Newton's formula

Let $f$ be integrable on $[a, b]$ and $F(x)$ be (one of) its primitive function on $(a, b)$. We have

$$\int_a^b f(x)\,\mathrm{d}x \;=\; [F(x)]_a^b \;=\; F(b) - F(a).$$

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**Example**

*Calculate* $\int_0^1 (2x + x^2)\,\mathrm{d}x$.

# Substitution

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$.
Let $\varphi$ be a real function differentiable on $(\alpha, \beta)$ such that $\varphi$ and $\varphi'$ are both continuous on $[\alpha, \beta]$.
Let $f$ be continuous on $[\varphi(\alpha), \varphi(\beta)]$ (or $[\varphi(\beta), \varphi(\alpha)]$).

If $f(\varphi(t))\, \varphi'(t)$ is integrable on $[\alpha, \beta]$, then

$$\int_{\alpha}^{\beta} f(\varphi(t))\, \varphi'(t)\, \mathrm{d}t = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, \mathrm{d}x.$$

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## Example

*Calculate* $\int_1^2 \frac{2\ln(t)^2}{t}\,\mathrm{d}t$.

# Integration by parts

Let $f$ and $g$ be differentiable on $(a, b)$ and let $f, g, f', g'$ be continuous on $[a, b]$. We have

$$\int_a^b f'(x)g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\,\mathrm{d}x.$$

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**Example**

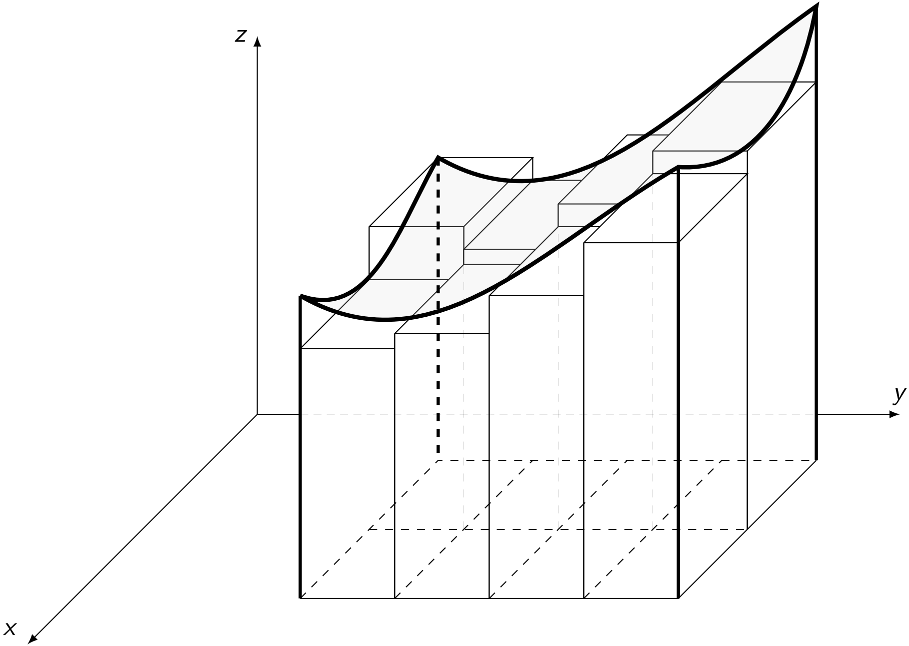
*Calculate* $\int_1^2 10x \ln x \,\mathrm{d}x$.

# Constrained Optimization, Multivariate Integration

# 2-variate function

Suppose we have a function $f : D \to \mathbb{R}$, where $D = [a, b] \times [c, d]$.

Imagine that this function represents (part of) a surface of some object.
What is the volume of this object?


z
y
x

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Let $\Delta_x = (x_i)_{i=0}^n$ define a partition of $[a, b]$ and $\Delta_y = (y_j)_{j=0}^m$ a partition of $[c, d]$. Then, $\Delta = \Delta_x \times \Delta_y$ defines a partitions of $D = [a, b] \times [c, d]$ into rectangles.

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Set

- $F_{\Delta,i,j} = \max\{f(x, y) \colon (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$ and
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while the lower Darboux sum of $f$ with respect to the partition $\Delta$ is

$$s_{f,\Delta} = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{\Delta,i,j}(x_i - x_{i-1})(y_j - y_{j-1}).$$

# Definition. . .

The upper Darboux integral (of $f$ over $D$) is

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If $D_f = d_f$, we call this value the (**double**) Darboux integral of $f$ over $D$, and denote it

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = D_f = d_f.$$

We say that $f$ is (Darboux-)integrable over $D$.

# How to calculate a double integral?

The following statement can be derived from the definition.

## Theorem

*If $f$ is integrable over $D = [a, b] \times [c, d]$ and one of the integrals*

$$\int_a^b \left( \int_c^d f(x, y) \, \mathrm{d}y \right) \mathrm{d}x \quad \textit{or} \quad \int_c^d \left( \int_a^b f(x, y) \, \mathrm{d}x \right) \mathrm{d}y$$

*exists, then it is equal to*

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## Example

*Calculate the double integral over $D = [0, 2] \times [-1, 2]$ of the function $f(x, y) = x^2 y + 1$.*

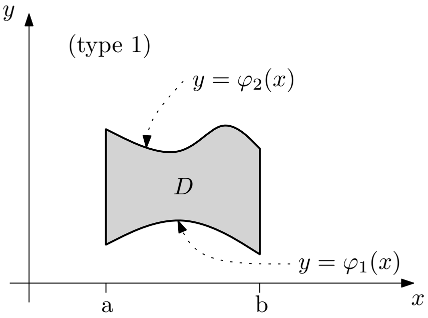
# And if $D$ is not a rectangle?

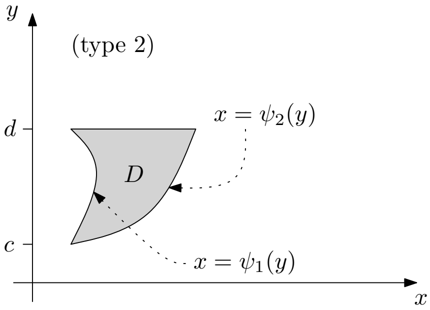
The definition is very similar: we approximate $D$ using smaller and smaller rectangular areas...

# Special types of domain $D$ (1/2)

We will consider the following two types of the domain $D$.

- (type 1) $x \in [a, b]$ and $y$ is bounded by continuous functions $\varphi_1(x)$ and $\varphi_2(x)$;
- (type 2) $y \in [c, d]$ and $x$ is bounded by continuous functions $\psi_1(y)$ and $\psi_2(y)$.



# Special types of domain $D$ (2/2)

Double integrals over such $D$ are calculated as follows.

## Theorem

*If the integral on the right side exists, then we have (for such a domain $D$):*

- *if $D$ is of type 1, then*

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\mathrm{d}y \right) \mathrm{d}x;$$

- *if $D$ is of type 2, then*

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\mathrm{d}x \right) \mathrm{d}y.$$

# Example

## Example

*Let* $D$ *be the region in the first quadrant between the lines* $x - 4y = 0$ *and* $x - 2y = 1$. *Calculate*

$$\iint_D \frac{x}{1+y^2}\,\mathrm{d}x\mathrm{d}y.$$