

# Mathematics for Informatics

General Algebra 2  
(lecture 5 of 12)

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# Outline

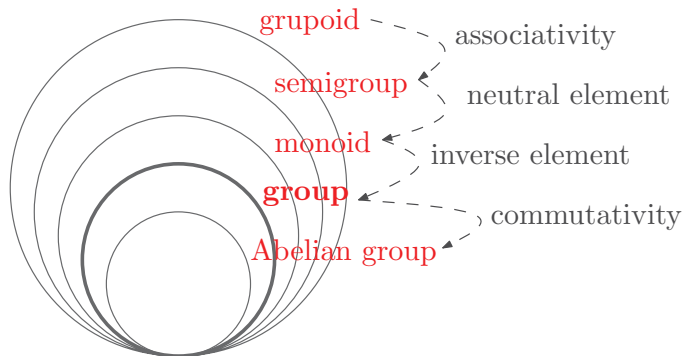
1 Reminder and Motivation

2 Subgroups

3 Cyclic groups

# Reminder of the last lecture

Hierarchy of structures of type “a set and a binary operation”



# Example (1/4)

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Let  $\mathbb{Z}_n = \{0, 1, 2, \dots, n - 1\}$  be the set of the residue classes modulo  $n$ .

The group  $(\mathbb{Z}_n, +_{(\pmod{n})})$  is the **additive group modulo  $n$** ; it is denoted by  $\mathbb{Z}_n^+$ .

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**Question (refined):** Which subset of  $\mathbb{Z}_{12}$  forms a group with the addition  $(\text{mod } 12)$ ?

**Answer:** There are quite a lot of them. To find out how to discover them, let us ask this subquestion:

*Sub-question:* Which is the smallest subset of  $\mathbb{Z}_{12}$  that forms a group with addition  $(\text{mod } 12)$  and contains the number 2?

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The wanted set is  $M = \{0, 2, 4, 6, 8, 10\}$ : we say that  $M$  is a **subgroup generated** by the set  $\{2\}$ .

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**Back to the original question:** there exist 6 different sets  $M \subseteq \mathbb{Z}_{12}$  such that  $(M, +(\text{mod } 12))$  is a group.

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# Definition of subgroup

## Definition

Let  $G = (M, \circ)$  be a group. A *subgroup* of the group  $G$  is a pair  $H = (N, \circ)$  such that:

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  - Similarly, we can define subgroupoids, subsemigroups, submonoids,...
  - A binary operation in the group  $G = (M, \circ)$  is considered to be a mapping from  $M \times M$  to  $M$ . The operation in a subgroup  $H = (N, \circ)$  is, to be precise, the restriction of this operation to the set  $N \times N$ .

# Trivial and proper subgroups

In each group  $G = (M, \circ)$ , there always exist at least two subgroups (if  $M$  contains only one element the two coincide):

- the group containing only the neutral element:  $(\{e\}, \circ)$ ,
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## Question

*If  $H$  is a subgroup of the group  $G$ , is the neutral element of  $H$  identical to the neutral element of  $G$ ?*

# Intersection of subgroups

## Theorem

Let  $H_1, H_2, \dots, H_n$ , with  $n \geq 1$ , be subgroups of a group  $G = (M, \circ)$ . Then

$$H' = \bigcap_{i=1,2,\dots,n} H_i$$

is also a subgroup of  $G$ .

# Power of an element

## Definition

Let  $G = (M, \circ)$  be a group with neutral element  $e$ . We define for each element  $a \in M$  and each positive  $n \in \mathbb{N}$  the  $n$ -th power of the element  $a$  as

$$\begin{aligned}
 a^0 &= e \\
 a^n &= \underbrace{a \circ a \circ \cdots \circ a}_{n \text{ times}} \\
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- $a^{n+m} = a^n \circ a^m$ ,
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For the additive notation of a group  $G = (M, +)$ , we define the  $n$ -th multiple of the element  $a$  and we denote it by  $n \times a$  (resp.  $-n \times a = n \times (-a)$ ).

# Order of a (sub)group

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The *order of a (sub)group*  $G = (M, \circ)$ , denoted  $|G|$ , is its number of elements. If  $M$  is an infinite set, the order is infinite. According to the order we distinguish between *finite* and *infinite groups*.

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## Example

The group  $\mathbb{Z}_{12}^+$  is of order 12. It has 6 subgroups:

- two trivial:  $\{0\}$  and  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ ;
- and four proper:  $\{0, 6\}$ ,  $\{0, 4, 8\}$ ,  $\{0, 3, 6, 9\}$ , and  $\{0, 2, 4, 6, 8, 10\}$ .

of order respectively 1, 2, 3, 4, 6 and 12.

# (Left) cosets of a subgroup

Let  $G$  be a group and  $H$  be one of its subgroups.

The (left) coset of  $H$  in  $G$  with respect to an element  $g \in G$  is the set

$$gH = \{gh : h \in H\} \quad (\text{or } g + H \text{ in additive notation})$$

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The index of  $H$  in  $G$ , denoted  $[G : H]$ , is the number of different cosets of  $H$  in  $G$ .

# Lagrange's Theorem

## Theorem

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## Question

Let  $G$  be a group of order  $n$  and  $k \in \mathbb{N}$  be such that  $k|n$ .  
Is there any subgroup of  $G$  of order  $k$ ?

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If for a set  $M$  it holds that  $\langle M \rangle = G$ , we say that  $M$  is a **generating set of  $G$** .

# Group generated by a set (2/2)

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The group  $\mathbb{Z}_{12}^+$  is generated, for instance, by the sets  $\{1\}$  and  $\{5\}$ , i.e.

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# Outline

- 1 Reminder and Motivation
- 2 Subgroups
- 3 Cyclic groups**

# Examples of groups generated by one element (1/2)

We have seen that the additive group  $\mathbb{Z}_{12}^+$  is equal to  $\langle 1 \rangle$ ,  $\langle 5 \rangle$ ,  $\langle 7 \rangle$ , and  $\langle 11 \rangle$ .

The following theorem says this is true in general:

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## Proof.

This statement is a consequence of a general theorem which will be proven later and of the fact that  $\langle 1 \rangle = \mathbb{Z}_n^+$  for all  $n \geq 2$ . □

# Examples of groups generated by one element (2/2)

The **multiplicative group modulo  $p$** , denoted  $\mathbb{Z}_p^\times$ , where  $p$  is a prime number, is the set  $\{1, 2, \dots, p-1\}$  with the operation of multiplication modulo  $p$ .

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Finding the generator(s) of a multiplicative group  $\mathbb{Z}_p^\times$  is more complicated than for an additive group  $\mathbb{Z}_n^+$ .

Multiplicative groups have more complicated and interesting structure.

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- $\mathbb{Z}_{11}^\times$  is cyclic, and  $2$  is a generator.

# Why “cyclic”?

Consider the multiplicative group  $\mathbb{Z}_{13}^\times$ .

Again, 2 is a generator. If we repeatedly compose the element 2 with itself we successively get all elements of the group:  $2^1 = 2$ ,  $2^2 = 4$ ,  $2^3 = 8$ ,  $2^4 = 3$ ,  $\dots$ ,  $2^{12} = 1$ .

The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.

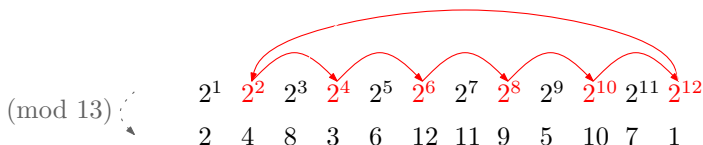
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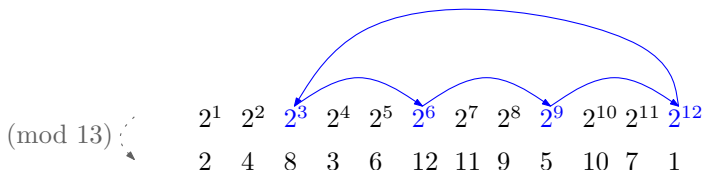


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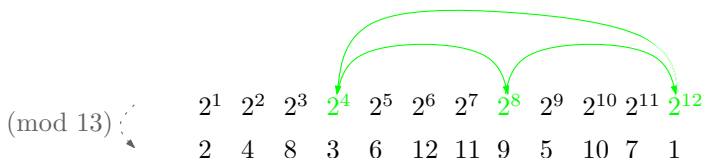
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# Fermat's Theorem (1/2)

## Theorem

In a cyclic group  $G = (M, \circ)$  of order  $n$ , for all elements  $a \in M$ , it holds that

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We have  $a^n = a^{qk} = (a^q)^k = e^k = e$ . □

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Applying the previous statement to  $\mathbb{Z}_p^\times$  we obtain the well-known **Fermat's Little Theorem**.

## Corollary (Fermat's Little Theorem)

*For an arbitrary prime number  $p$  and an arbitrary  $1 \leq a < p$  we have that*

$$a^{p-1} \equiv 1 \pmod{p}.$$

# How to find all generators (1/2)

Generally, to find all generators is not an easy task (e.g., in groups  $\mathbb{Z}_p^\times$  we are not able to do it algorithmically); but if we have one, it is easy to find all the others.

## Theorem

*If  $(G, \circ)$  is a cyclic group of order  $n$  and  $a$  is one of its generator, then  $a^k$  is a generator if and only if  $k$  and  $n$  are coprime.*

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To prove the Theorem we use the following

## Lemma

*Let  $D = \{mk + \ell n \mid m, \ell \in \mathbb{Z}\}$  and  $d = \min\{|a| \mid a \in D \setminus \{0\}\}$ .  
Then  $d = \gcd(k, n)$ .*

# How to find all generators (2/2)

## Corollary

*In a cyclic group of order  $n$ , the number of all generators is equal to  $\varphi(n)$ .*

Where  $\varphi$  is the **Euler's (totient) function**, which assigns to each integer  $n$  the number of integers less than  $n$  that are coprime with  $n$

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An effective algorithm for evaluating  $\varphi(n)$  does not exist; if it existed, we would be able to find the integer factorization of arbitrarily large  $n$  and RSA would not be safe!



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Let  $G = \langle a \rangle$  be a cyclic group and  $H$  one of its proper subgroup.

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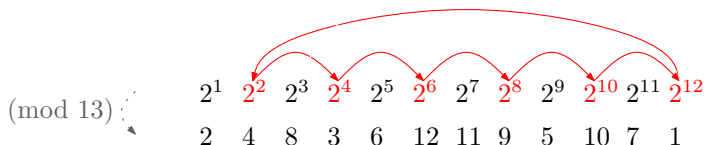
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# Illustration of validity previous statements

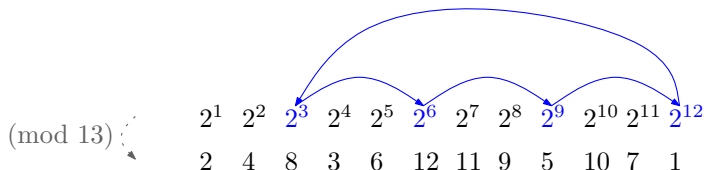
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## Example

*Find the order of all elements in  $\mathbb{Z}_5^\times$  and in  $\mathbb{Z}_7^\times$ .*