MIE-MPI: Tutorial 4

`created: October 25, 2019, 15:57`

Comments: The goal of this tutorial is to understand the notions groupoid, semigroup, monoid and group: given a pair of a set and an operation on it, you should be able to correctly verify the relevant properties.

4.1 Algebra

Exercise 4.1. Find out which of the following groupoids are groups and which are Abelian groups.

1. $(\mathbb{N}, +)$;
2. $(\mathbb{Z}, +)$;
3. $(\mathbb{R}, \cdot)$;
4. $(\mathbb{C}, +)$.

Exercise 4.2. Which of following couples form a group?

1. $(\mathbb{Z}^-, +)$;
2. $(\mathbb{Q}, +)$;
3. $(\mathbb{Q}_0^+, +)$;
4. $(\{-1, 1\}, +)$;
5. $(\{-1, 1\}, \cdot)$.

Exercise 4.3. Which of the following sets with the operation $\circ$ defined for all a, b by

$$a \circ b = a^b$$

form a groupoid/semigroup/group/Abelian group?

1. $(\mathbb{Q}, \circ)$;
2. $(\mathbb{N}, \circ)$;
3. $(\{-1, 1\}, \circ)$.

Exercise 4.4. Which of the following sets of complex square matrices create a group with the common matrix multiplication?

1. real matrices;
2. regular matrices;
3. regular diagonal matrices;
4. upper (lower) regular triangular matrices.

Exercise 4.5. Let us consider the set $M = \mathbb{Z} \times \mathbb{Z} \times \{1, -1\}$ and the operation $\otimes$ on it defined by:

$$(k_1, k_2, 1) \otimes (\ell_1, \ell_2, \varepsilon) = (k_1 + \ell_1, k_2 + \ell_2, \varepsilon)$$
$$(k_1, k_2, -1) \otimes (\ell_1, \ell_2, \varepsilon) = (k_1 + \ell_1, k_2 + \ell_2, -\varepsilon)$$

for $\varepsilon \in \{1, -1\}$ is a group.

Exercise 4.6. Let us define the operations $\sqcap$ and $\triangle$ on $\mathbb{R}$ as:

$$a \sqcap b = a + b + 1 \quad \text{and} \quad a \triangle b = a + b + ab$$

for arbitrary $a, b \in \mathbb{R}$.

Prove that

- $(\mathbb{R}, \sqcap)$ is an Abelian group;
- $(\mathbb{R}, \triangle)$ is not a group.

Exercise 4.7. Let ρ be a plane. For arbitrary points $A, B \in \rho$ let $A \star B$ be the central point of the segment AB. Is $(\rho, \star)$ a group?

Exercise 4.8. Let for arbitrary $a, b \in \mathbb{N}$ be $a \circ b = a^b$ (see Exercise 4.3). Find all triples $a, b, c \in \mathbb{N}$ for which we have $(a \circ b) \circ c = a \circ (b \circ c)$.

Exercise 4.9. Let us consider the set $\mathbb{N}$ and the operation $\wedge$ such that for arbitrary $a, b \in \mathbb{N}$ we have $a \wedge b$ equal to the greatest common divisor of a, b. Is $(N, \wedge)$ a group?

Exercise 4.10. Decide whether following table is Cayley table of a group?

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

How can we recognize that it is Cayley table of an Abelian group?