Linear Algebra with Application

(LAWA 2021)

Lecture 10

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As in the previous lecture, let us consider matrices over $\mathbb{R}$.

1 Elementary operations and determinants

Because of its recursive definition, it is often hard to compute the determinant. Using elementary operations we can create more zeros in a matrix, but these operations will change the determinant. Let us see how.

Theorem 1 *Let $A \in \mathcal{M}_{n,n}(\mathbb{R})$ be square matrix.*

(1) If B is obtained from A by interchanging two different rows (elementary row operation of type i)) or two different columns (elementary column operation of type i)), then

$$det(B) = -det(A).$$

(2) If B is obtained from A by multiplying a row (elementary row operation of type ii)) or a column (elementary column operation of type ii)) by a number k, then

$$det(B) = k \cdot det(A).$$

(3) If B is obtained from A by adding a multiple of some row of A to a different row (elementary row operation of type iii)) or a multiple of some column of A to a different column (elementary column operation of type iii)), then

$$det(B) = det(A).$$

The proof of the previous theorem is not hard, but its out of the scope of this course.

Example 2 Let us consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 4 & 7 & 0 \end{pmatrix}.$$

Because of point (3) of Theorem 1, we know that the determinant does not change if we subtract twice the 1-row from the 3-row and the 2-row from the 3-row, so

$$\det(A) = \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 2 & 3 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, using Laplace expansion along the third row we have

$$\det(A) = -1 \cdot (-1)^{3+3} \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = -(1 \cdot 3 - 2 \cdot 2) = 1.$$

Example 3 Following the previous result we can compute the determinant of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 4 & 7 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{pmatrix}$$

(`Exercise`).

2 Some properties on determinants

The following result easily follows from point (2) of Theorem 1.

Theorem 4 *Let $A \in \mathcal{M}_{n,n}(\mathbb{R})$. Then for any number $k \in \mathbb{R}$*

$$det(kA) = k^n det(A).$$

Example 5 Let us consider the matrix

$$A = \begin{pmatrix} 2 & 4 \\ 6 & 2 \end{pmatrix} = 2 \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

Then

$$\det(A) = 4 - 24 = -20 = 2^2 \cdot (1 - 6).$$

The determinant of diagonal and triangular matrices is quite easy to compute.

Theorem 6 *If a square matrix is triangular, then its determinant is the product of the entries of the main diagonal.*

Proof.[Idea] Let us consider an upper triangular matrix $A \in \mathcal{M}_{n,n}(\mathbb{R})$, that is a matrix of the form

$$A = \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 & 0 \\ * & a_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & a_{n-1,n-1} & 0 \\ * & * & \cdots & * & a_{n,n} \end{pmatrix}$$

where the $*$ represent arbitrary real numbers. Then, by considering recursively the Laplace expansion along the first row, we find that

$$\det(A) = a_{1,1}a_{1,2}\cdots a_{n,n}.$$

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Example 7 Let us consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & -1 & 7 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

The matrix is (lower) triangular. Then, by Theorem 6 we have

$$\det(A) = 1 \cdot (-1) \cdot 2 \cdot (-3) = 6.$$

Column operations from a matrix A to a matrix B can be accomplished by doing the corresponding row operations from A^T to B^T, then take the transpose to B^T back to B. The following theorem tell us that transposing a matrix does not change its determinatn.

Theorem 8 *Let $A \in \mathcal{M}_{n,n}(\mathbb{R})$. Then*

$$det\left(A^T\right) = det(A).$$

Example 9 Let us consider a matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

and its transpose

$$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Then

$$\det\left(A^T\right) = 1 \cdot 4 - 3 \cdot 2 = -2 = 1 \cdot 4 - 2 \cdot 3 = \det(A).$$

Theorem 10 *Let E be an elementary matrix.*

1. *If E is of type $i)$ then $det(E) = -1$.*
2. *If E is of type $ii)$ and is obtained from I by multiplying a row (or a column) by a number k, then $det(E) = k$.*
3. *If E is of type $iii)$, then $det(E) = 1$.*

Proof. In this proof we will consider elementary matrices obtained from I_n, with $n \in \mathbb{N}$, using a row elementary operation. The case of matrices obtained by elementary column operations can be proved in a symmetric way.

1. Let us first consider the case of an elementary matrix of type $i)$. Let i, j with $1 \leq i < j \leq n$ (the case $j < i$ is proved symmetrically) be such that $I \xrightarrow[R_1 \leftrightarrow R_j]{i)} E$. Let us prove our result by induction on n. If $n = 2$, then necessarly we

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\det(E) = -1$.

If $n > 2$ then, we have

$$E = \begin{array}{cc} \begin{array}{cccccccc} 1 & & & i & & j & & n \end{array} & \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} & \begin{array}{c} 1 \\ \\ \\ i \\ \\ j \\ \\ n \end{array} \end{array}$$

Let $k \neq i, j$. Then the (k, k)-entry of the matrix is 1 while for all $h \neq k$, the (k, h)-entry of E is 0 (that is, the k-row has zeros everywhere except at its k-th position). Using Laplace Expansion Theorem, we can consider the cofactor expansion of E along the k-row and obtain

$$\begin{aligned} \det(E) &= (-1)^{k+k} \cdot 1 \cdot \det(A_{k,k}) + \sum_{h \neq k} (-1)^{k+h} \cdot 0 \cdot \det(A_{k,h}) \\ &= \det(A_{k,k}), \end{aligned}$$

where $A_{k,k}$ is of the form

$$A_{k,k} = \begin{array}{cc} \begin{array}{cccccccc} 1 & & & i' & & j' & & n-1 \end{array} & \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} & \begin{array}{c} 1 \\ \\ \\ i' \\ \\ j' \\ \\ n-1 \end{array} \end{array}$$

for certains $1 \leq i' < j' \leq n-1$ (resp. $j' < i'$). Note that $A_{k,k}$ is an elmentary matrix of size $(n-1) \times (n-1)$ and that it can be obtain from I_{n-1} by the elementary row-operation

$$I_{n-1} \xrightarrow[R_{i'} \leftrightarrow R_{j'}]{i)} A_{k,k}.$$

By inductive hypothesis $\det(A_{k,k}) = 1$. Thus $\det(E) = 1$ as well.

2. Let us now consider the case of an elementary matrix of type $ii)$. Let i and k, with $1 \leq i \leq n$ and $k \neq 0$ be such that $I \xrightarrow[R_i \to kR_i]{ii)} E$. Then, we can prove that $\det(E) = k$ (`Exercise`).

3. Let us finally consider the case of an elementary matrix of type $iii)$. Let i, j and k with $1 \leq i < j \leq n$ (resp. $j < i$) and $k \in \mathbb{R}$ be such that $I \xrightarrow[R_j \to R_j + kR_i]{ii)} E$. Then, we can prove that $\det(E) = 1$ (`Exercise`).

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Example 11 Let us consider the three elementary matrices

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

. We have

$$\det(E_1) = -1, \quad \det(E_2) = \frac{1}{3} \quad \text{and} \quad \det(E_3) = 1.$$

The following important result tell us how to compute the determinant of the product of two matrices.

Theorem 12 (Product Theorem) *Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$. Then*

$$det(AB) = det(A)\, det(B).$$

Proof. Let us first consider an elementary matrix $E \in \mathcal{M}_{n,n}(\mathbb{R})$. We know, from Lecture 8 that the matrix $B = EA$ represents the matrix A after we apply an elementary row operation and that the matrix $C = AE$ represents the matrix A after we apply an elementary column operation. Combining Theorem 10 with Theorem 9 of the previous lecture, we have that:

- If E is of type i), then $\det(E) = -1$. Moreover, B is obtained from A by interchanging two different rows and the matrix C is obtained from A by interchanging two columns. Thus

$$\det(EA) = \det(B) = -\det(A) = \det(E)\det(A)$$

  and

$$\det(AE) = \det(C) = -\det(A) = \det(A)\det(E).$$

- If E is of type ii), then $\det(E) = k$ for a certain number $k \neq 0$. The matrix B is obtained from A by multiplying all elements of a certain row by k, while the matrix C is obtained from A by multiplying all elements of a certain column by k. Thus

$$\det(EA) = \det(B) = k \cdot \det(A) = \det(E)\det(A)$$

  and

$$\det(AE) = \det(C) = k \cdot \det(A) = \det(A)\det(E)$$

- If E is of type iii), then $\det(E) = 1$. Moreover B is obtained from A by adding a multiple of some row of A to a different row, while C is obtained from A by adding a multiple of some column of A t a differenc column. Thus

$$\det(EA) = \det(B) = \det(A) = \det(E)\det(A)$$

  and

$$\det(AE) = \det(C) = \det(A) = \det(A)\det(E).$$

By Theorem 2 of Lecture 9, we know that there exists two invertible matrices $U, V \in \mathcal{M}_{n,n}(\mathbb{R})$ such that

$$UAV = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix},$$

with $r = \operatorname{rank}(A)$. Moreover, from wat we have seen in Lecture 9, we know that there exists elementary matrices $E_1, E_2, \ldots, E_p, \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_q$ such that

$$U = E_1 E_2 \cdots E_p \quad \text{and} \quad V = \widetilde{E}_1 \widetilde{E}_2 \cdots \widetilde{E}_q.$$

Thus one has

$$\begin{aligned} A &= U^{-1} \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} V^{-1} \\ &= E_p^{-1} \cdots E_2^{-1} E_1^{-1} \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \widetilde{E}_q^{-1} \cdots \widetilde{E}_2^{-1} \widetilde{E}_1^{-1}. \end{aligned}$$

Note that $E_1^{-1}, E_2^{-1}, \ldots, E_p^{-1}, \widetilde{E}_1^{-1}, \widetilde{E}_2^{-1}, \ldots, \widetilde{E}_q^{-1}$ are also elementary matrices. If $\operatorname{rank}(A) < n$, then

$$\begin{aligned}
\det(AB) &= \det\left(E_p^{-1}\cdots E_2^{-1}E_1^{-1}\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}\widetilde{E}_q^{-1}\cdots\widetilde{E}_2^{-1}\widetilde{E}_1^{-1}B\right) \\
&= \det\left(E_p^{-1}\right)\cdots\det\left(E_2^{-1}\right)\det\left(E_1^{-1}\right)\det\left(\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}\widetilde{E}_q^{-1}\cdots\widetilde{E}_2^{-1}\widetilde{E}_1^{-1}B\right) \\
&= \det\left(E_p^{-1}\right)\cdots\det\left(E_2^{-1}\right)\det\left(E_1^{-1}\right)\cdot 0 \\
&= 0,
\end{aligned}$$

where the third equality follows from Corollary 11 in Lecture 9 and the fact that in the matrix

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}\widetilde{E}_q^{-1}\cdots\widetilde{E}_2^{-1}\widetilde{E}_1^{-1}B$$

the last $n-r$ rows (and the last $n-r$ columns) are zero rows (zero columns).

If $\operatorname{rank}(A) = n$, then

$$\begin{aligned}
\det(AB) &= \det\left(E_p^{-1}\cdots E_2^{-1}E_1^{-1}I_n\widetilde{E}_q^{-1}\cdots\widetilde{E}_2^{-1}\widetilde{E}_1^{-1}B\right) \\
&= \det\left(E_p^{-1}\right)\cdots\det\left(E_1^{-1}\right)\det(I_n)\det\left(\widetilde{E}_q^{-1}\right)\cdots\det\left(\widetilde{E}_1^{-1}\right)\det(B) \\
&= \det\left(E_p^{-1}\cdots E_2^{-1}E_1^{-1}I_n\widetilde{E}_q^{-1}\cdots\widetilde{E}_2^{-1}\widetilde{E}_1^{-1}\right)\det(B) \\
&= \det(A)\det(B).
\end{aligned}$$

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Example 13 Let us consider the two matrices

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Then

$$\det(AB) = \det\begin{pmatrix} 7 & 8 \\ 4 & 1 \end{pmatrix} = -25 = -5 \cdot 5 = \det(A)\det(B)$$

We give the following result without proof. However the second point is easy to prove (`Exercise`).

Theorem 14 *Let $A \in \mathcal{M}_{n,n}(\mathbb{R})$. Then*

1. *A is invertible if and only if $det(A) \neq 0$.*
2. *If A is invertible, then $det\left(A^{-1}\right) = \frac{1}{det(A)}$.*

Example 15 Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$ and let us suppose that

$$\det(A) = 2 \quad \text{and} \quad \det(B) = -3.$$

Using the previous theorems we can compute $\det\left(2A^3B^{-1}A^TB^2\right)$ (`Exercise`).