Linear Algebra with Application

(LAWA 2021)

Lecture 11

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1 Block triangular matrices

Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$ and $B \in \mathcal{M}_{p,q}(\mathbb{R})$ be two matrices. We call *block upper triangular matrix* a matrix of the form

$$\begin{pmatrix} A & X \\ O & B \end{pmatrix},$$

where X is is a matrix of size $m \times q$ and $O = O_{p,n}$. A *blok lower triangular matrix* will be defined in a symmetrical way.

Theorem 1 *Let A and B be two square matrices, possibly of different size. Then*

$$\det \begin{pmatrix} A & X \\ O & B \end{pmatrix} = det(A)\, det(B)$$

and

$$\det \begin{pmatrix} A & O \\ X & B \end{pmatrix} = det(A)\, det(B)\,.$$

Proof. Let us consider the block upper triangular matrix. We can decompose it as

$$\begin{pmatrix} A & X \\ O & B \end{pmatrix} = \begin{pmatrix} I & O \\ O & B \end{pmatrix} \begin{pmatrix} A & X \\ O & I \end{pmatrix}.$$

From the Product Theorem we know that

$$\det \begin{pmatrix} A & X \\ O & B \end{pmatrix} = \det \begin{pmatrix} I & O \\ O & B \end{pmatrix} \det \begin{pmatrix} A & X \\ O & I \end{pmatrix}.$$

By repeated cofactor expansions it is easy to see that

$$\det \begin{pmatrix} I & O \\ O & B \end{pmatrix} = \det (B) \quad \text{and} \quad \det \begin{pmatrix} A & X \\ O & I \end{pmatrix} = \det (A).$$

To prove the result for the block lower triangular matrices we can take the transpose and use Theorem 8 in Lecture 10. ∎

Example 2 Let

$$A = \begin{pmatrix} 1 & 2 & a & b & c \\ 2 & 5 & d & e & f \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

We have

$$\det (A) = \det \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \det \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 1 \cdot 8 = 8.$$

2 Adjacent of a matrix

Let $A \in \mathcal{M}_{n,n} (\mathbb{R})$. We define the *adjoint* of A the transpose of the matrix of cofactors:

$$\operatorname{adj} (A) = \left(C_{i,j}(A)\right)^T.$$

Note that $\operatorname{adj} (A)$ is also a $n \times n$-matrix. When it is clear from the context, we will write $C_{i,j}$ instead of $C_{i,j}(A)$.

Example 3 Let us consider the matrix

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 4 & 7 & 3 \\ -2 & 8 & 5 \end{pmatrix}.$$

Its adjoint is the matrix:

$$\begin{aligned}\operatorname{adj}(A) &= \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}^T \\ &= \begin{pmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ C_{1,3} & C_{2,3} & C_{3,3} \end{pmatrix} \\ &= \begin{pmatrix} 11 & -8 & 7 \\ -26 & 13 & -13 \\ 46 & -24 & 21 \end{pmatrix}.\end{aligned}$$

Example 4 Let us consider a generic 2×2-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Its adjoint is the matrix

$$\operatorname{adj}(A) = \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}^T = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Using the Laplace Expansion Theorem, we can obtain the following result.

Theorem 5 (Adjoint Formula) *Let A be a square matrix. Then*

1. $A \cdot adj(A) = det(A)\, I = adj(A) \cdot A$.
2. *If* $det(A) \neq 0$, *then* $A^{-1} = \frac{1}{det(A)} adj(A)$.

Example 6 Let us consider the matrix

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 4 & 7 & 3 \\ -2 & 8 & 5 \end{pmatrix}.$$

Its determinant is

$$\det(A) = 3 \det \begin{pmatrix} 7 & 3 \\ 8 & 5 \end{pmatrix} - \det \begin{pmatrix} 4 & 7 \\ -2 & 8 \end{pmatrix} = 3 \cdot 11 - 1 \cdot 46 = -13,$$

and its adjoint is given in Example 3. Thus the inverse of A is the matrix

$$A^{-1} = -\frac{1}{13} \begin{pmatrix} 11 & -8 & 7 \\ -26 & 13 & -13 \\ 46 & -24 & 21 \end{pmatrix}.$$

3 Cramer's Rule

If A is a $n \times n$-matrix and B is a $n \times 1$-column, then let us set $A_i(B)$ the $n \times n$-matrix obtained from A by replacing the i-column by B.

Example 7 Let us consider again the matrix A given in Example 3 and the column $B = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$. Then we have

$$A_1(B) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 7 & 3 \\ 1 & 8 & 5 \end{pmatrix}, \quad A_2(B) = \begin{pmatrix} 3 & 1 & -1 \\ 4 & 0 & 3 \\ -2 & 1 & 5 \end{pmatrix} \quad \text{and} \quad A_3(B) = \begin{pmatrix} 3 & 0 & 1 \\ 4 & 7 & 0 \\ -2 & 8 & 1 \end{pmatrix}.$$

Theorem 8 (Cramer's Rule) *Let us consider the system of linear equations $AX = B$, where A is invertible. Then, the unique solution is given by*

$$X^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix}$$

where

$$x_i^* = \frac{det(A_i(B))}{det(A)} \quad \text{for each } i = 1, 2, \ldots, n.$$

Example 9 Let us consider the system of linear equations

$$\begin{cases} 5x_1 - 7x_2 + 8x_3 & = & 23 \\ 2x_1 + 6x_2 - 9x_3 & = & 61 \\ -x_1 - 4x_2 + 3x_3 & = & 19 \end{cases}.$$

The matrix

$$A = \begin{pmatrix} 5 & -7 & 8 \\ 2 & 6 & -9 \\ -1 & -4 & 3 \end{pmatrix}$$

is invertible since its determinant is

$$\det(A) = -127 \neq 0.$$

To find x_2 let us consider the matrix

$$A_2(B) = \begin{pmatrix} 5 & 23 & 8 \\ 2 & 61 & -9 \\ -1 & -19 & 3 \end{pmatrix}.$$

Since $\det(A_2(B)) = 313$, we have

$$x_2 = \frac{\det(A_2(B))}{\det(A)} = -\frac{313}{127}.$$

Example 10 Let us consider the system of linear equations

$$\begin{cases} 3x_1 - x_3 &= 1 \\ 4x_1 + 7x_2 + 3x_3 &= 0 \\ -2x_1 + 8x_2 + 5x_3 &= 1 \end{cases}.$$

Using the Cramer's Rule we can find the solution (`Exercise`)

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

4 Diagonalization

Let us recall from Lecture 3 that a $n \times n$-matrix D is called a *diagonal matrix* if all its entries off the main diagonal are zeros; that is, if D has the form

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are numbers. These numbers are not necessarly reals (we will see an example later), so, instead that in $\mathbb{R}$, let us suppose that $\lambda_i \in \mathbb{C}$ for all i.

A n-square matrix A is called *diagonalizable* if there exists an invertible matrix $P \in \mathcal{M}_{n,n}(\mathbb{R})$ such that $P^{-1}AP = D$ is diagonal. In this case, the invertible matrix P is called a *diagonalizing matrix* for A.

Diagonalization is one of the most important ideas inlinear algebra. One of its uses is to give us an efficient method to calculate powers $A, A^2, A^3, \ldots$ of a square matrix A.

Theorem 11 *Let A be a diagonalizable matrix. Let us suppose that P is a diagonalizing matrix and $D = P^{-1}AP$. Then, for any $k \in \mathbb{N}$ one has*

$$A^k = PD^kP^{-1}.$$

Proof. (`Exercise`) ∎