Linear Algebra with Application

(LAWA 2021)

Lecture 2

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In this lecture we will introduce matrices over $\mathbb{R}$. In the next lectures we will consider also the case of matrices over different sets of numbers.

1 Definition of matrix

A *matrix* A over $\mathbb{R}$ is a rectangular array of real numbers of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

with $m, n \in \mathbb{N}$ and $a_{ij} \in \mathbb{R}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

The (i,j)-*entry* of A is the number a_{ij}, while its i-*row* and its j-*column* are respectively

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

A matrix having m rows and n columns is called a $m \times n$-matrix. When $m = n$, the matrix is called a n-*square matrix* (or simply *square matrix*).

Example 1 The matrix

$$A = \begin{pmatrix} 1 & -2 & 0 & \pi \\ 0 & 2 & -3 & 0 \\ \sqrt{5} & -1 & 0 & 7 \\ 2 & 6 & 7 & 9 \end{pmatrix}$$

is a square matrix of size 4×4, that is it has 4 rows and 4 columns.

The matrix

$$B = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

is a 2×4-matrix.

The $(3, 2)$-entry of A is -1 and the $(1, 4)$-entry of B is 2. The 2-row of A is $\begin{pmatrix} 0 & 2 & -3 & 0 \end{pmatrix}$, while the 1-column of B is $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

We say that two matrices A and B are *equal*, and we write $A = B$, if and only if A and B have the same size and the corresponding entries are equal.

The set of matrices over $\mathbb{R}$ of size $m \times n$ is denoted by $\mathcal{M}_{m,n}(\mathbb{R})$, that is

$$\mathcal{M}_{m,n}(\mathbb{R}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \mid a_{ij} \in \mathbb{R} \text{ for all } 1 \leq i \leq m,\ 1 \leq j \leq n \right\}.$$

Sometimes, when it is clear from the context, we can also denote the matrix A by (a_{ij}).

2 Matrix addition

Given two matrices $A = (a_{ij})$ and $B = (b_{ij})$ having the same size we can add them just by adding the corresponding entries, that is the *sum* of A and B is the matrix

$$A + B = (a_{ij} + b_{ij}).$$

. Similarly, we can define the *difference*

$$A - B = (a_{ij} - b_{ij}).$$

The *negative* of A is the matrix $-A$ obtained as

$$-A = (-a_{ij}).$$

Example 2 Let un consider the tree matrices

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 5 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ -2 & 3 & 3 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}.$$

We have

$$A+B = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 5 & 3 \end{pmatrix}, \quad A-B = \begin{pmatrix} 1 & -1 & -4 \\ 7 & -1 & -3 \end{pmatrix} \quad \text{and} \quad -A = \begin{pmatrix} -1 & 0 & 2 \\ -5 & -2 & 0 \end{pmatrix}.$$

We can not define $A + C$ or $A - C$ since A and C have different size.

Let us denote by $O_{m,n}$ the $m \times n$-matrix having each entry equal to 0. Such a matrix is called the *zero matrix* of size $m \times n$. When $m = n$ we will simply write O_n instead of $O_{n,n}$. When the size is clear from the context, it is simply denoted by O.

Example 3 The zero matrices of order 2×3 and 2×2 are respectively

$$O_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R}) \quad \text{and} \quad O_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}).$$

Theorem 4 *The set $\mathcal{M}_{m,n}(\mathbb{R})$ with the sum operation is an Abelian group.*

Proof. (`Exercise`) ∎

3 Matrix multiplication

Let $n \in \mathbb{N}$. Let us consider a n-row-matrix, that is a matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \in \mathcal{M}_{1,n}(\mathbb{R})$$

and an n-column-matrix, that is a matrix

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{R}).$$

We define the *dot product* of A and B as the number

$$\sum_{k=1}^{n} a_k b_k = a_1 b_1 + a_2 b_2 + \cdots a_n b_n \in \mathbb{R}.$$

The *product* of a $m \times n$-matrix A and an $n \times p$-matrix B is the $m \times p$-matrix AB having as (i,j)-entry the dot product of the i-row of A and the j-column of B. So, if the two matrices are

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{np} & \end{pmatrix}$$

then the (i,j)-entry of AB is

$$\sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

and the matrix AB has the form

$$AB = \begin{pmatrix} \sum_{k=1}^{n} a_{1k}b_{k1} & \sum_{k=1}^{n} a_{1k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{1k}b_{kp} \\ \sum_{k=1}^{n} a_{2k}b_{k1} & \sum_{k=1}^{n} a_{2k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{2k}b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}b_{k1} & \sum_{k=1}^{n} a_{mk}b_{k2} & \cdots & \sum_{k=1}^{n} a_{mk}b_{kp} \end{pmatrix}.$$

Note that the product AB is defined if and only if the number of columns of A equals the number of columns of B.

Example 5 Let $A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$, $B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$ and $C = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$. Then we have

$$AB = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}, \qquad BA = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$
$$BC = \begin{pmatrix} -1 & -1 & 4 \\ 2 & 4 & -2 \end{pmatrix}, \quad AC = \begin{pmatrix} 2 & 5 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

We can not define CA because of the size.

Proposition 6 *The matrix multiplication is associative, i.e., that for all matrices A, B, C having the right sizes, we have $(AB)C = A(BC)$.*

Proof. (`Exercise`) ∎

The $n \times n$ *identity matrix* is the matrix $I_n \in \mathcal{M}_{n,n}(\mathbb{R})$ with 1s on the *main diagonal*, i.e., the entries of the form (i,i), and zero elsewhere. When the size is clear from the context, it is simply denoted by I.

Example 7 One has

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 8 *Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$. Then*

$$I_m A = A \quad \text{and} \quad AI_n = A.$$

Proof. (`Exercise`) ∎

Combining Propositions 6 and 8 we can easily show the following result.

Theorem 9 *The set $\mathcal{M}_{n,n}(\mathbb{R})$ with the product operation is a monoid.*

Proof. (`Exercise`) ∎

Let us now consider the set of matrices with both the sum and the product operations.

Proposition 10 *The distributive laws hold both for the sum and for the subtraction, i.e., for all matrices A, B, C having the right sizes, we have*

- $A(B + C) = AB + AC$;
- $A(B - C) = AB - AC$;
- $(A + B)C = AC + BC$;
- $(A - B)C = AC - BC$.

Proof. (`Exercise`) ∎

Corollary 11 *The set of n-square matrices over $\mathbb{R}$, i.e., $\mathcal{M}_{n,n}(\mathbb{R})$ with the matrix addition and the matrix multiplication is a ring.*

When it is clear from the context, we will use the same notation for the set and the ring of matrices, i.e., we will write $\mathcal{M}_{n,n}(\mathbb{R})$ instead of $(\mathcal{M}_{n,n}(\mathbb{R}), +, \cdot)$.

Example 12 Let A, B as in Example 5. We have

$$AB = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = BA$$

The previous example shows that commutativity does not hold in general for matrices.

Corollary 13 *The ring $\mathcal{M}_{n,n}(\mathbb{R})$ is not a commutative ring.*

Example 14 Let us consider the matrix $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. One has

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The last example shows that the ring $\mathcal{M}_{n,n}(\mathbb{R})$ has zero divisors.

Corollary 15 *The ring $\mathcal{M}_{n,n}(\mathbb{R})$ is not an integral domain (nor a field).*

4 Scalar multiplication

Let $A = (a_{ij}) \in \mathcal{M}_{m,n}(\mathbb{R})$ be a matrix and $\lambda \in \mathbb{R}$ a real number. The *scalar product* λA it defined as the matrix of the form (λa_{ij}), that is

$$\lambda A = \lambda \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}.$$

Example 16 Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 3 & -3 \end{pmatrix}$. Then

$$2A = \begin{pmatrix} 2 & 0 \\ -2 & 4 \\ 6 & -6 \end{pmatrix}.$$

Proposition 17 *Let A, B be two matrices of the same size and let λ, μ be two real numbers. Then*

1. $\lambda(A + B) = \lambda A + \lambda B$;
2. $(\lambda + \mu)A = \lambda A + \mu A$;
3. $\lambda(\mu A) = (\lambda\mu)A$;
4. $1A = A$;
5. $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

Proof. (`Exercise`) ∎