Linear Algebra with Application

(LAWA 2021)

Lecture 3

Francesco Dolce

francesco.dolce@fjfi.cvut.cz

March 3rd, 2021

As in the previous lecture, let us consider matrices over the field $\mathbb{R}$.

1 Transposition

Let $A = (a_{ij}) \in \mathcal{M}_{m,n}(\mathbb{R})$ be a matrix. The *transpose* of A is the matrix $A^T \in \mathcal{M}_{n,m}(\mathbb{R})$ defined as

$$A^T = (b_{ij}) \quad \text{with} \quad b_{ij} = a_{ji} \text{ for all } i, j.$$

A matrix A is called *symmetric* if $A^T = A$.

Example 1 Let

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -2 \\ 5 & -1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have

$$A^T = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 5 \\ 0 & -1 \\ -2 & 0 \end{pmatrix} \quad \text{and} \quad C^T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Moreover, the matrix C is symmetric, while A and B are not.

Proposition 2 *Let A, B be two matrices over $\mathbb{R}$ having the right size and let $\lambda \in \mathbb{R}$. Then*

1. *If A is symmetric then $m = n$;*
2. $(A^T)^T = A$;
3. $(\lambda A)^T = \lambda A^T$;
4. $(A + B)^T = A^T + B^T$;
5. $(AB)^T = B^T A^T$.

Proof. (`Exercise`) ■

2 Matrix inverse

Let us consider a square matrix $A \in \mathcal{M}_{n,n}(\mathbb{R})$. A matrix B is called an *inverse* of A if $AB = I$ and $BA = I$. Note that if such a matrix B exists, then it has the same size as A.

A square matrix having an inverse is called an *invertible matrix.*

Example 3 Let us consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$. The matrix A is invertible and the matrix $B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ one of its inverses. Indeed,

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

and

$$BA = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Similarly to what we have done in Lecture 1 (Proposition 9) we can prove that the inverse of a matrix, when it exists, is unique.

Proposition 4 *Let $A \in \mathcal{M}_{n,n}(\mathbb{R})$ an invertible matrix. Then its inverse is unique.*

Proof. Let $B, C \in \mathcal{M}_{n,n}(\mathbb{R})$ and let us suppose that both matrices are inverses of A. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

■

Whenever a matrix A is invertible we denote by A^{-1} its unique inverse.

Note that not all matrices have inverses.

Example 5 Let us conside the matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. This matrix is not invertible. Indeed, if we suppose by contradiction that there exists a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AB = I$, then we would have

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

which implies at the same time that $a = 1$ and $a = 0$, a contradiction.

Also, non-square matrices do not have inverses and the zero matrix O_n is not invertible neither (`Exercise`).

Let us now give some properties about inverses.

Theorem 6 *The following properties hold for square matrices.*

1. *The identity matrix I is invertible and its inverse is I itself.*
2. *If A is invertible, then so is A^{-1}, and $\left(A^{-1}\right)^{-1} = A$.*
3. *If A and B are invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
4. *Let $A_1, A_2, \ldots A_k$ be invertible matrices, then their product $A_1A_2 \cdots A_k$ is also invertible, and $(A_1A_2 \cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_1^{-1}$*
5. *If A is invertible, then for all $k \geq 1$ the matrix A^k is invertible as well and $(A^k)^{-1} = (A^{-1})^k$.*
6. *If A is invertible, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.*
7. *If A^T is invertible, then so is A and $A^{-1} = ((A^T)^{-1})^T$.*
8. *If A is invertible and $\lambda \neq 0$ is a real number, then λA is also invertible and $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$.*

Proof.

1. This easily follows from the fact that $II = I$.
2. The second item also follows from the fact that $A^{-1}A = I$ and $AA^{-1} = I$.
3. It holds because

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

4. Let us prove it by induction on k. The case $k = 1$ is trivial and the case $k = 2$ follows from point 3. So, let us suppose that the property holds for $k - 1$, that is that $A_1 A_2 \cdots A_{k-1}$ is invertible and that its inverse is $A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$. Then

$$(A_1 A_2 \cdots A_{k-1} A_k) = (A_1 A_2 \cdots A_{k-1}) A_k$$

is a product of two invertible matrices and, by the previous point, is invertible itself. Moreover, its inverse is exactly

$$A_k^{-1} (A_1 A_2 \cdots A_{k-1})^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}.$$

5. This easily follows from the previous item by chosing $A_i = A$ for all $1 \le i \le k$.

6. Using the last point of Proposition 2 we have

$$A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I$$

and

$$(A^{-1})^T A^T = (A A^{-1})^T = I^T = I.$$

7. Using both the fact that $(A^T)^{-1} = (A^{-1})^T$, proved in the previous point, and that $(A^T)^T = A$, proved in Proposition 2, we have

$$A((A^T)^{-1})^T = A((A^{-1})^T)^T = A A^{-1} = I$$

and

$$((A^T)^{-1})^T A = ((A^{-1})^T)^T A = A^{-1} A = I.$$

8. To prove the last point we use Proposition 17 in Lecture 2 and show that

$$(\lambda A) \left(\frac{1}{\lambda} A^{-1} \right) = \left(\lambda \frac{1}{\lambda} \right) \left(A A^{-1} \right) = 1 I = I$$

and

$$\left(\frac{1}{\lambda} A^{-1} \right) (\lambda A) = \left(\frac{1}{\lambda} \lambda \right) \left(A^{-1} A \right) = 1 I = I.$$

■

Corollary 7 *Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$. If A and AB are both invertible, then B is also invertible.*

Proof. `(Exercise)` ■

3 Diagonal and triangular matrices

A square matrix $A = (a_{ij}) \in \mathcal{M}_{n,n}(\mathbb{R})$ is called *diagonal* if every entry not in the main diagonal is 0, that is if for every i, j with $1 \leq i, j \leq n$ and $i \neq j$ one has $(a_{ij}) = 0$.

Example 8 The matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

are diagonal.

When a matrix $A = (a_{ij})$ is diagonal, we can also denote it simply as $A = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.

Example 9 Let us consider again the three matrices of Example 8. We have

$$A = \operatorname{diag}(1, 2, -5), \quad B = \operatorname{diag}(-1, 4, 0, 1) \quad \text{and} \quad C = \operatorname{diag}(0, 0).$$

Proposition 10 *Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$ be two diagonal matrices. Then*

1. $A + B$ is diagonal;

2. AB is diagonal.

Proof. (`Exercise`) ∎

A square matrix $A = (a_{ij})$ is called *upper triangular* if every entry below the main diagonal is zero, that if for every i, j with $i > j$ one has $a_{ij} = 0$. An upper triangular matrix is called *strictly upper triangular* if the entries on the main diagonal are zero as well.

In a symmetric way we define *lower triangular* and *strictly lower triangular* matrices.

Example 11 Let us consider the four matrices

$$A = \begin{pmatrix} 6 & 9 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 7 & 6 & 0 \end{pmatrix}.$$

The matrices A and B are upper triangular. Moreover the matrix B is strictly upper triangular. Simarly C and D are lower triangular with D being strictly lower triangular.

Proposition 12 *Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$ be two upper triangular matrices. Then*

1. *$A + B$ is upper triangular.*

2. *AB is upper triangular.*

Proof. (`Exercise`) ■

A similar result also holds by replacing the condition "upper triangular" with "strictly upper triangular", "lower triangular" or "strictly lower triangular".