Linear Algebra with Application
(LAWA 2021)

Lecture 4

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In this lecture we introduce linear equations and systems of linear equations. We also discuss how to use matrices to represent and solve such equations.

1 Variables, coefficients and solutions

We call a *linear equation* in the *variables* $x_1, x_2, \ldots, x_n$, with $n \in \mathbb{N}$, an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b, \tag{1}$$

where $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$. The numbers $a_1, a_2, \ldots, a_n$ are called the *coefficients* of the variables $x_1, x_2, \ldots, x_n$, while the number b is called the *constant term* of the equation.

We can represent the coefficients using the row

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

called the *coefficient row* of the equation, and the variables using the column matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

called the *matrix of variables* of the equation.

A column

$$X_0 = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$$

is called a *solution* of the linear Equation (1) if

$$A \cdot X_0 = a_1 s_1 + a_2 s_2 + \cdots + a_n s_n = b$$

that is, if replacing x_i with s_i for every $1 \leq i \leq n$ on the right side, we obtain b.

Example 1 Let us consider the linear equation

$$2x_1 + x_2 - x_3 = 3$$

The coefficients of the variables x_1, x_2, x_3 are respectively $2, 1$ and -1, while the constant term of the equation is 3. The coefficient row is

$$A = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}$$

and a solution of the linear equation is

$$X_0 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$$

since

$$2 \cdot 1 + 1 \cdot 1 - 1 \cdot 0 = 3.$$

Another possible solution for the equation is $X_1 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$.

From the previous example, we see that the solution of a linear equation, in general, is not unique. We call a possible solution a *particular solution* of the equation. A way to find all solutions of an equation is given by fixing two variables and compute the third one with respect to the previous.

Example 2 Let us consider the equation in Example 1. By setting $x_1 = s$ and $x_2 = t$, we find that $2s + t - x_3 = 3$, which implies that $x_3 = 2s + t - 3$. So, all solutions have the form

$$X = \begin{pmatrix} s & t & 2s + t - 3 \end{pmatrix}^T$$

for certain $s, t \in \mathbb{R}$.

Following the terminology of the previous example, we call X the *general solution* of the equation and s, t the *parameters* of the solution.

2 Systems of linear equations

A *system of linear equations* is a finite collection of linear equations. Its general form is

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{cases}, \tag{2}$$

where $x_1, x_2, \ldots, x_n$ are variables and the coefficients $a_{i,j}$ and the constant terms b_i, with $1 \leq i \leq m$ and $1 \leq j \leq n$, are in $\mathbb{R}$. When we need more precision, we call the previous system a system of m equations in n variables.

A solution satisfying every equation of a system is called a *solution of the system*.

Example 3 The system of two linear equations in the variabels x, y

$$\begin{cases} x + y & = & 10 \\ 2x - y & = & 5 \end{cases} \tag{3}$$

has an unique solution $X_0 = \begin{pmatrix} 5 & 5 \end{pmatrix}^T$.

Note that some system may have no solution. In this case we say that the system is *inconsistent*.

Example 4 The system

$$\begin{cases} x + y & = & 1 \\ 2x + 2y & = & 3 \end{cases} \tag{4}$$

has no solution (`Exercise`), so it is inconsistent.

A system with at least one solution is called *consistent*.

Example 5 The system

$$\begin{cases} x + y + z & = & 2 \\ x - y + z & = & 0 \end{cases} \tag{5}$$

has infintely many solution (`Exercise`). Thus it is consistent.

Let us now show how we can represent a system of m equations in n variables, like the one in Equation 2, using matrices. The *coefficient matrix* and the *constant matrix* for this system are respectively the $m \times n$-matrix and the m-column defined as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

We can also combine the two to obtain the *augmented matrix* defined as the $m \times (n+1)$-matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}.$$

Example 6 Let us consider the system (3). Its matrix of variable is $X = \begin{pmatrix} x \\ y \end{pmatrix}$ while its constant matrix and its coefficient matrix are respectively

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 10 \\ 5 \end{pmatrix}.$$

Its augmented matrix is the (2×3)-matrix

$$\begin{pmatrix} 1 & 1 & 10 \\ 2 & -1 & 5 \end{pmatrix}.$$

Note that, using the coefficient matrix, the variable matrix and the constant matrix, we can represent the system of linear equations (2) as a single matrix equation

$$AX = B. \tag{6}$$

Example 7 Using Example 6 we can represent the system (3) as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}.$$

Example 8 Les us consider the system of three linear equations in four variables

$$\begin{cases} x_1 - x_4 + x_3 & = & 5 \\ x_3 - 3x_4 & = & -1 \\ x_4 & = & 2 \end{cases}. \tag{7}$$

The previous system is in a very special form and can be solved by using *back-substitution.*

- From the last equation we get $x_4 = 2$.
- Then we substitute 2 for the variable x_4 into the second last equation to solve for $x_3 = 5$.
- We substitute $x_3 = 5$ and $x_4 = 2$ and obtain $x_1 = 2$.
- Finally we replace x_2 with a parameter s.

The *general solution* of the original system has thus the form

$$X = \begin{pmatrix} 4s & s & 5 & 2 \end{pmatrix}^T,$$

where s is the parameter of the solution. That means that every solution can be obtained by replacing s with a real number.

The system of linear equations presented in Example 8 was in a very special form. In the next sections we will see how to use this technique to solve a system in a more general form.

3 Equivalent systems

Two systems of linear equations having the same set of solutions are called *equivalent*.

Example 9 Let us consider the system in Example 5 and let us swap the two equations:

$$\begin{cases} x - y + z &= 0 \\ x + y + z &= 2 \end{cases}. \tag{8}$$

It is clear that this new system is equivalent to the system (5).

Example 10 Starting again from the system in Example 5, let us multiply the left and the right term of the second equation by 2:

$$\begin{cases} x + y + z &= 2 \\ 2x - 2y + 2z &= 0 \end{cases}. \tag{9}$$

One can see that this system is also equivalent to the system (5). (`Exercise`)

Example 11 Using one more time the system in Example 5, let us replace the second equation by the sum of the two original equations.

$$\begin{cases} x + y + z &= 2 \\ 2x + 2z &= 2 \end{cases}. \tag{10}$$

Also in this case it can be shown that the system is equivalent to (5). (`Exercise`)

Following the previous examples we define the three *elementary operations* on a system of linear equation as:

i) interchange two equations;

ii) multiply one of the equations by a nonzero number;

iii) add a multiple of one equation to a different equation.

A similar set of operations can be defined also on matrices. We call *elementary row operations* on a matrix the following operations:

i) interchange two rows;

ii) multiply one of the rows by a nonzero number;

iii) add a multiple of one row to a different row.

These row operations can be seen as the counterpart of the elementary operations on system on the related augmented matrices.

Example 12 Let us consider the system of linear equations (5). Its augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

The augmented matrices of the systems (8), (9) and (10) are respectively:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & -2 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 \end{pmatrix}.$$

They are obtained by the first matrix by applying an elementary row operation of type, respectively, *i*), *ii*) and *iii*). Indeed one has

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{i)} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow[R_2 \to 2R_2]{ii)} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & -2 & 2 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow[R_2 \to R_1 + R_2]{iii)} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 \end{pmatrix}.$$

Theorem 13 *Let us consider a system of linear equations. The system obtained by applying an elementary operation is equivalent to the original system.*

The previous theorem tells us that in order to find the solution of a system we can apply a series of elementary operations to reduce this system to one which is easier to solve.