Linear Algebra with Application
(LAWA 2021)
Lecture 5

Francesco Dolce
francesco.dolce@fjfi.cvut.cz

March 17, 2021

In this lecture we will see how to find the solutions of a general system of linear equations using the elementary operations we introduced in the previous lecture.

1 Gaussian elimination

Before giving the algorithm let us start with an example.

Example 1 Let us consider the following system of linear equations

$$\left\{\begin{array}{rcl} -x_3+3x_4 & = & 1 \\ x_1-4x_2+x_3 & = & 5 \\ 2x_1-8x_2+2x_3-3x_4 & = & 4 \end{array}\right. . \tag{1}$$

The augmented matrix of the system (1) is

$$\begin{pmatrix} 0 & 0 & -1 & 3 & 1 \\ 1 & -4 & 1 & 0 & 5 \\ 2 & -8 & 2 & -3 & 4 \end{pmatrix}.$$

Starting fom the 1-row and proceeding down, we find the first column from the left containing a non-zero entry. In our case this is is the first column, since the $(2,1)$-entry is 1. Using an elementary row operation of type i) let us move the the 2-row on the top (that is let us interchange the 2-row with the 1-row). We get the matrix

$$\begin{pmatrix} 1 & -4 & 1 & 0 & 5 \\ 0 & 0 & -1 & 3 & 1 \\ 2 & -8 & 2 & -3 & 4 \end{pmatrix}.$$

The first non-zero entry, that is the $(1,1)$-entry 1, in the 1-row is called the *leading 1* for the first row. Using an elementary row operation of type iii) we can subtracte 2 times the 1-row from the 3-row and obtain the matrix

$$\begin{pmatrix} 1 & -4 & 1 & 0 & 5 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & -3 & -6 \end{pmatrix}.$$

This way we have "cleared" the first column below the first leading 1.

Now, let us forget the first row and let us modify the rest of the matrix in a similar way we have done so far, starting from the second row. That is, starting from the 2-row and proceeding down we find the first column from the left containing a non-zero entry, in our case the $(2,3)$-entry -1. Since this non-zero element is already in the good position, we don't need to interchange rows (elementary row operation of type i)). On the other hand, to obtain a leading 1 in the second row, we can use an elementary row operation of type ii) and multiply the whole 2-row by -1. This way we obtain the matrix

$$\begin{pmatrix} 1 & -4 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 & -6 \end{pmatrix}.$$

Since below this second leading 1 we only have zeros, we don't need to "clear" the bottom part of the column.

Now that we have the two first leading ones, let us do the same operation starting from the third row. The first non-zero entry from the left is the $(3,4)$-entry -3. As before, let us use an elementary row operation of type ii) and multiply the 3-row by $-\frac{1}{3}$ in order to create the leading 1 in the third column. We thus obtain the matrix

$$\begin{pmatrix} 1 & -4 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{2}$$

The matrix (2) is the augmented matrix of the system of linear equations

$$\begin{cases} x_1 - x_4 + x_3 & = & 5 \\ x_3 - 3x_4 & = & -1 \\ x_4 & = & 2 \end{cases}. \tag{3}$$

From what we have seen in the previouslecture, that means that the system of linear equations in Equation (1) is equivalent to the system of linear equations in Equation (3). Thus the two have the same set of solutions.

Matrices like the ones in Equation (1) have a special name. A matrix is said to be in *row-echelon form*, and it will be called a *row-echelon matrix*, if the following conditions are satisfied:

1. All zero rows are at the bottom;

2. The first non-zero entry from the left in each non-zero row is a 1, and we call it the *leading 1* of the row;

3. Each leading 1 is to the right of all leading 1s in the rows above it.

A row-echelon matrix is said to be in *reduced row-echelon form*, or is it called a *reduced row-echelon matrix*, if it satisfies as well the condition:

4. Each leading 1 is the only non-zero entry in its column.

Example 2 The matrix in Equation (2) has is in row-echelon form but non in reduced row-echelon form. All the other matrices in Example 1 are not in row-echelon form.

The following matrices are in reduced row-echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The following result shown that given a system of linear equations we can always find an equivalent *easier* system which can be solved using the back-substitution (when a solution exists). The algorithm in the proof is called the *Gaussian Algorithm*

Theorem 3 *Every matrix can be carried, in a finite number of steps to a row-echelon form (reduced, if desiderd), using a sequence of elementary row operations.*

Proof. [**Gaussian Algorithm**] Do the following steps until you obtain a (reduced) row-echelon matrix.

> **Step 1-1** Starting from the 1-row, find the first non-zero entry in the first column from the left and interchange the corresponding row with the 1-row;
>
> **Step 1-2** Multiply the 1-row by a costant to create the first leading 1 in the 1-row;

Step 1-3 Subtract multiples of the 1-row from lower rows to obtain all zeros below the leading 1.

Step 1-3b [to obtain a reduced row-echelon matrix] Subtract multiples of the 1-row from upper rows to obtain all zeros above the leading 1.

Step 2-1 Starting from the 2-row, find the first non-zero entry in the first column from the left and interchange the corresponding row with the 2-row;

Step 2-2 Multiply the 2-row by a costant to create the first leading 1 in the 2-row;

Step 2-3 Subtract multiples of the 2-row from lower rows to obtain all zeros below the leading 1.

Step 2-3b [to obtain a reduced row-echelon matrix] Subtract multiples of the 2-row from upper rows to obtain all zeros above the leading 1.

$\vdots$

Step k**-1** Starting from the k-row, find the first non-zero entry in the first column from the left and interchange the corresponding row with the k-row;

Step k**-2** Multiply the k-row by a costant to create the first leading 1 in the k-row;

Step k**-3** Subtract multiples of the k-row from lower rows to obtain all zeros below the leading 1.

Step k**-3b** [to obtain a reduced row-echelon matrix] Subtract multiples of the k-row from upper rows to obtain all zeros above the leading 1.

■

The algorithm presented in the proof of Theorem 3 can be implemented in your favorite program language (`Exercise`).

Note that the way to carry a matrix to its reduced form is not unique. Indeed one can obtain the same result by changing the order of some of the steps. That means that, even though the algorithm always work, it is not necessarily the most efficient one.

Example 4 Let us prove that the following system of linear equations has no solution

$$\begin{cases} x+y &= 1 \\ x-2 &= 2 \\ y+z &= 1 \end{cases} \tag{4}$$

The reduction of the augmented matrix is

$$\begin{pmatrix}1 & 1 & 0 & 1\\ 1 & 0 & -1 & 2\\ 0 & 1 & -1 & 1\end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{iii)} \begin{pmatrix}1 & 1 & 0 & 1\\ 0 & -1 & 1 & 1\\ 0 & 1 & -1 & 1\end{pmatrix}$$
$$\xrightarrow[R_2 \to -R_2]{ii)} \begin{pmatrix}1 & 1 & 0 & 1\\ 0 & 1 & -1 & -1\\ 0 & 1 & -1 & 1\end{pmatrix}$$
$$\xrightarrow[R_3 \to R_3 - R_2]{iii)} \begin{pmatrix}1 & 1 & 0 & 1\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 2\end{pmatrix}$$
$$\xrightarrow[R_3 \to \frac{1}{2}R_3]{ii)} \begin{pmatrix}1 & 1 & 0 & 1\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 1\end{pmatrix},$$

where at every passage we show which elementary row operation we are applying: for instance $\xrightarrow[R_2 \to R_2 - R_1]{iii)}$ means that we are using the elementary row operation of type $iii)$ by subtracting once the first row to the second row.

The last row of the last matrix corresponds to the equation

$$0x + 0y + 0z = 1,$$

which is clearly never satisfied, no matter the choice of x, y and z. Since the solution of the system must satisfy all equations, the system of linear equation corresponding to this augmented matrix, and thus the equivalent original system, has no solution.

Example 5 Let us find all solutions of the following system of linear equations

$$\begin{cases} x + y - z &= 3\\ -2x - y &= -4\\ 4x + 2y + 3z &= -1 \end{cases}.$$

(`Exercise`)

Looking at the row-echelon reduction of the augmented matrix of a system of linear equations, we can also determine if the original system has no solution, a unique solution or infinitely many solution. Indeed, let us suppose that we have a system of m linear equations in n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases} \tag{5}$$

If we reduce A to a row-echelon form R, then we have the following cases

1. If there is a leading 1 in the last column, then the system of linear equation has no solution (as seen in Example 4);

2. If there is no leading 1 in the last column, then the system has at least one solution. We call the number of leading 1s the *rank* of the matrix, and we denote it by $\mathrm{rank}\,(A)$. Note that the rank does not change under elementary row operations, so $\mathrm{rank}\,(A) = \mathrm{rank}\,(R)$. Moreover, since there are no leading 1s in the last column, we have $\mathrm{rank}\,(A) \leq n$. We can thus distinguish two cases:

   (a) If there is at least one solution and $\mathrm{rank}\,(A) = n$, then the solution is unique, and it can be found simply by back-substitution.

   (b) If there is at least one solution and $\mathrm{rank}\,(A) < n$, then the system has infinitely many solutions (as in Example 1). In this case we assign $n - \mathrm{rank}\,(A)$ parameters to the variable corresponding to the columns without leading 1s, and we solve, again, by back-substitution (see also Example 8 in Lecture 4).

This method is also called *Gaussian elimination.*

We can summarize what seen so far in the following theorem.

Theorem 6 *For any system of linear equations there are exactly three possibilities:*

1. *The system has no solution.*

2. *The system has a unique solution.*

3. *The system has infinitely many solutions. Moreover, if n is the number of variables and r is the rank of the augmented matrix, then the set of solutions has exactly $(n - r)$ parameters.*

When it is clear from the context we will also call *rank* of the system the rank of the associated augmented matrix.