Linear Algebra with Application

(LAWA 2021)

Lecture 6

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In this lecture we will consider the special case of a system of linear equations where all the constant terms are zero. We will see that this extra condition garantuees the presence of at least one solution.

1 Homogeneous systems

Let us consider a system of linear equations in which all the constant terms are zero, such as the following one

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{cases} \tag{1}$$

Such a system is called *homogeneous*. It is clear that when choosing $x_1 = x_2 = \cdots = x_n = 0$ the equations are satisfied. Thus a homogeneous system has always (at least) one solution, namely $X_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}^T$. We call this solution the *trivial* solution of the homogeneous system. Any other possible solution, that is if at least one of the entries is non-zero, is called a *non-trivial* solution.

From what seen in the previous lecture we can prove the following result.

Theorem 1 *If a homogeneous system of linear equations has more variables than equations, then it has nontrivial solutions.*

Proof. Let us consider a system of m linear equations in n variables and let us suppose that $n > m$. Let A be the augmented matrix of the system. We know that the system has at least one solution, the trivial one. Since rank $(A) \leq m < n$, it follows from Theorem 6 of Lecture 5, that the system has infinitely many solutions. ■

Example 2 Let us consider the following homogeneous system

$$\begin{cases} x_1 - 2x_2 + 4x_3 - x_4 + 5x_6 &= 0 \\ -2x_1 + 4x_2 - 7x_3 + x_4 + 2x_5 - 8x_6 &= 0 \\ 3x_1 - 6x_2 + 12x_3 - 3x_4 + x_5 + 15x_6 &= 0 \\ 2x_1 - 4x_2 + 9x_3 - 3x_4 + 3x_5 + 12x_6 &= 0 \end{cases}.$$

The augmented matrix of this system is

$$\begin{pmatrix} 1 & -2 & 4 & -1 & 0 & 5 & 0 \\ -2 & 4 & -7 & 1 & 2 & -8 & 0 \\ 3 & -6 & 12 & -3 & 1 & 15 & 0 \\ 2 & -4 & 9 & -3 & 3 & 12 & 0 \end{pmatrix}.$$

The $(1,1)$-entry being the first leading 1, we proceed as in the previous lecture to clean the rest of the 1-column and, after that, to find the other leading 1s; we continue with elementary row operations until we obtain the reduced row-echelon matrix (`Exercise`)

$$\begin{pmatrix} 1 & -2 & 0 & 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the original system is equivalent to the following one:

$$\begin{cases} x_1 - 2x_2 + 3x_4 - 3x_6 &= 0 \\ x_3 - x_4 + 2x_6 &= 0 \\ x_5 &= 0 \end{cases}.$$

The leading 1s in the augmented matrix correspond to the variables x_1, x_3 and x_5, and the rank of the system is 3. The other variables, i.e., x_2, x_4 and x_6 are called *non-leading variables.* To find the general solution we will associate some parameters, let us call them s, t and u, to the non-leading variables. The general solution has thus the form

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2s - 3t + 3u \\ s \\ t - 2u \\ t \\ 0 \\ u \end{pmatrix} = s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let us denote

$$X_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We call X_1, X_2 and X_3 *basic solutions* of the system. The general solution X is a *linear combination* of the three basic solutions, since

$$X = sX_1 + tX_2 + uX_3$$

for any choice of s, t and u.

We can generalize the previous example in the following theorem.

Theorem 3 *Let us consider a system of homogeneous linear equations in n variables and let us suppose that its rank is r. Then*

- *The Gaussian algorithm produces exactly $n - r$ basic solutions;*
- *Every solution is a linear combination of these basic solutions.*

Given a general system of linear equations we can associate to it a homogeneous system by replacing the constant terms with zero. We will refer to this system as the *associated homogeneous system* of the original one.

Example 4 Let us consider the following system of 3 linear equations in 4 variables

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 &= 2 \\ -x_1 + 2x_2 + x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 &= 1 \end{cases}. \tag{2}$$

A possible solution of the system 2 is

$$X_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Note that this particular solution is not, in general, unique and it is not always easy to find.

As seen in a previous lecture, we can rewrite this sytem of linear equations as a single matrix equation

$$AX = B$$

where

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

are respectively the coefficient matrix, the matrix of variables and the matrix of constants of the system.

The associated homogeneous system is represented by the matrix equation

$$\begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3}$$

By reduction of the augmented matrix of the homogeneous system in a reduced row-echelon form

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1}]{iii)} \begin{pmatrix} 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 - R_2 \\ R_3 \to R_3 - +2R_2}]{iii)} \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we find that the general solution of the system (3) is

$$X' = s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

where s and t are parameters representing arbitrary numbers.

The general solution of the system (2) is

$$X = X_0 + X' = X_0 + sX_1 + tX_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix},$$

with s and t arbitrary numbers, X_0 the particular solution seen before and X_1, X_2 the basic solutions computed above.

The previous example illustrate the following theorem.

Theorem 5 *Let us consider the system of linear equations $AX = B$, and let us suppose that X_0 is a particular solution. Then*

1. *if X' is a solution to the associated homogeneous system $AX = O$, then $X = X_0 + X'$ is a solution to the system $AX = B$.*

2. *Every solution to the system $AX = B$ has the form $X = X_0 + X'$ for some solution X' to the associated homogeneous system $AX = O$.*

Example 6 Let us consider the system of linear equations

$$\left\{\begin{array}{rcl} x_1 - 2x_2 + 2x_3 - x_4 & = & 1 \\ 2x_1 - 4x_2 + 3x_3 + x_4 & = & 2 \\ 3x_1 - 6x_2 + 5x_3 & = & 3 \end{array}\right. .$$

Using the Gaussian elimination and Theorem 5, we can write the general solution to the system as the sum of a particular solution and the general solution to the associated homogeneous system (`Exercise`).