Linear Algebra with Application

(LAWA 2021)

# Lecture 7

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In this lecture, as well as in the following ones, we will consider matrices over $\mathbb{R}$. However most of the results can be generalized using any field (like $\mathbb{Q}$ or $\mathbb{C}$).

## 1 The Matrix Inverse algorithm

In Lecture 3 we defined the inverse of a square matrix $A$ as the matrix $B$ such that

$$AB = I \quad \text{and} \quad BA = I,$$

where $B$ has the same size of $A$ and $I$ is the identity matrix.

**Example 1** Let us consider the matrix

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}).$$

Let us prove that the inverse of $A$ is its square $A^2$. Indeed, the matrix $A^2$ is given by

$$A^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

while the matrix $A^3$ is

$$A^3 = A^2 A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Thus $AA^2 = A^3 = I$ and $A^2 A = A^3 = I$, which prove the claim.

Finding the inverse of a given square matrix, when this exists, is not generally a trivial task. In the following example we show how to use the tools from the previous lectures in order to find the inverse of a matrix.

**Example 2** Let us consider the $2 \times 2$ matrix

$$A = \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix}$$

and let us suppose that its inverse exists and has the form

$$B = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

for certains $x_1, x_2, x_3, x_4 \in \mathbb{R}$.

Since $BA = I$, we have

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 & -5x_1 + 2x_2 \\ 2x_3 + x_4 & -5x_3 + 2x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is equivalent to the system of four linear equations in four variables

$$\left\{ \begin{array}{rcl} 2x_1 + x_2 & = & 1 \\ -5x_1 + 2x_2 & = & 0 \\ 2x_3 + x_4 & = & 0 \\ -5x_3 + 2x_4 & = & 1 \end{array} \right. . \tag{1}$$

Using the Gaussian algorithm on the augmented matrix of the system (1), we can find the equivalent matrix in reduced row-echelon form as follows:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 1 \\ -5 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -5 & 2 & 1 \end{pmatrix} \xrightarrow[R_1 \to \frac{1}{2}R_1]{ii)} \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -5 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -5 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 + 5R_1]{iii)} \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{9}{2} & 0 & 0 & \frac{5}{2} \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -5 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \to \frac{2}{9}R_2]{ii)} \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -5 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 - \frac{1}{2}R_2]{iii)} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2}{9} \\ 0 & 1 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -5 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3 \to \frac{1}{2}R_3]{ii)} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2}{9} \\ 0 & 1 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -5 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 + 5R_3]{iii)} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2}{9} \\ 0 & 1 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{9}{2} & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 \to \frac{2}{9}R_4]{ii)} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2}{9} \\ 0 & 1 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{9} \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 - \frac{1}{2}R_4]{iii)} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2}{2} \\ 0 & 1 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 1 & 0 & -\frac{1}{9} \\ 0 & 0 & 0 & 1 & \frac{2}{9} \end{pmatrix}.$$

Thus we get the solution

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} \frac{2}{9} & \frac{5}{9} & -\frac{1}{9} & \frac{2}{9} \end{pmatrix}^T.$$

One can check that considering the system of linear equations associated to the matrix equation

$$AB = \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - 5x_3 & 2x_2 - 5x_4 \\ x_1 + 2x_3 & x_2 + 2x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

one find the same solution (`Exercise`).

The (unique) inverse of $A$ is thus the matrix

$$B = \begin{pmatrix} \frac{2}{9} & \frac{5}{9} \\ -\frac{1}{9} & \frac{2}{9} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 2 & 5 \\ -1 & 2 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}).$$

The following result gives us a method to compute the inverse of an invertible matrix using the Gaussian Algorithm.

**Theorem 3 (Matrix Inverse Algorithm)** *Let $A$ be a square matrix. If there exists a sequence of elementary row operations that carry $A \to I$, then $A$ is invertible and this same sequence carries $I \to A^{-1}$. Thus, applying the same sequence of row operations on the matrix $\begin{pmatrix} A & I \end{pmatrix}$, one has the reduction*

$$\begin{pmatrix} A & I \end{pmatrix} \to \begin{pmatrix} I & A^{-1} \end{pmatrix}.$$

**Example 4** Let $A$ be the matrix defined in Example 2. The reduction to the reduced row-echelon form of the matrix $\begin{pmatrix} A & I \end{pmatrix}$ is the following

$$\begin{aligned}
\begin{pmatrix} 2 & -5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} & \xrightarrow[R_1 \leftrightarrow R_2]{i)} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -5 & 1 & 0 \end{pmatrix} \\
& \xrightarrow[R_2 \to R_2 - 2R_1]{iii)} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -9 & 1 & -2 \end{pmatrix} \\
& \xrightarrow[R_2 \to -\frac{1}{9}R_2]{ii)} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{9} & \frac{2}{9} \end{pmatrix} \\
& \xrightarrow[R_1 \to R_1 - 2R_2]{iii)} \begin{pmatrix} 1 & 0 & \frac{2}{9} & \frac{5}{9} \\ 0 & 1 & -\frac{1}{9} & \frac{2}{9} \end{pmatrix}
\end{aligned}$$

Thus the inverse of the matrix $\begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix}$ is the matrix $\begin{pmatrix} \frac{2}{9} & \frac{5}{9} \\ -\frac{1}{9} & \frac{2}{9} \end{pmatrix}$ (that is consistent with what we have seen in Example 2).

Using Theorem 3 we can find a formula to the inverse of an invertible $2 \times 2$-matrix.

**Example 5** Let us consider the $2 \times 2$-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$. Then $A$ is invertible and its inverse is the matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Indeed, by using the algorithm described in Theorem 3 we have the reduction (`Exercise`)

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{d}{\Delta} & -\frac{b}{\Delta} \\ 0 & 1 & -\frac{c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$$

where $\Delta = ad - bc$ is called the *determinant* of $A$. We also call the matrix

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

the *adjoint* of $A$ (we will discuss more about determinants and adjoints later).

To double check we can also verify that

$$\begin{aligned} AA^{-1} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \tfrac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \tfrac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & -cb+ad \end{pmatrix} \\ &= I \end{aligned}$$

and

$$\begin{aligned} A^{-1}A &= \tfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \tfrac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{pmatrix} \\ &= I. \end{aligned}$$

**Example 6** Let us consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -5 \\ 4 & 1 & 1 \end{pmatrix}.$$

Using the matrix inverse algorithm we can show that (`Exercise`) the inverse of $A$ is the matrix

$$B = \frac{1}{26} \begin{pmatrix} -8 & 3 & 7 \\ 22 & -5 & -3 \\ 10 & -7 & 1 \end{pmatrix}.$$

To make sure that the answer is right it is enough to verify that $AB = I$ and $BA = I$.

If a matrix $A$ is not invertible, then no sequence of row operations can carry $A \to I$. Hence the algorithm breaks down because a row of zeros is encountered.

**Example 7** The matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 4 & 7 & 1 \end{pmatrix}$$

has no inverse. Indeed, let us try the matrix inverse algorithm on $A$.

$$\begin{pmatrix} 1 & 2 & -1 & & 1 & 0 & 0 \\ 2 & 3 & 3 & & 0 & 1 & 0 \\ 4 & 7 & 1 & & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1}]{iii)} \begin{pmatrix} 1 & 2 & -1 & & 1 & 0 & 0 \\ 0 & -1 & 5 & & -2 & 1 & 0 \\ 0 & -1 & 5 & & -4 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \to -R_2]{ii)} \begin{pmatrix} 1 & 2 & -1 & & 1 & 0 & 0 \\ 0 & 1 & -5 & & 2 & -1 & 0 \\ 0 & -1 & 5 & & -4 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + R_2}]{iii)} \begin{pmatrix} 1 & 0 & 9 & & -3 & 2 & 0 \\ 0 & 1 & -5 & & 2 & -1 & 0 \\ 0 & 0 & 0 & & -2 & -1 & 1 \end{pmatrix}.$$

Since $A$ will never be transformed to the identity matrix by elementary row operations, $A$ is not invertible.

# 2 Inverses and systems of linear equations

As we have seen in Lecture 5, some systems of linear equations have a unique solution. Here we show how to find such a solution.

**Theorem 8** *Let us consider a system of $n$ linear equations in $n$ variables and let us suppose that we can write this system in matrix form as*

$$AX = B.$$

*If the $n$-square matrix $A$ is invertible, the system has the unique solution*

$$X = A^{-1}B.$$

*Proof.* Note that, since the system of linear equations has $n$ equations and $n$ variables, then $A$ has size $n \times n$. Since $A^{-1}$ is well defined, and $A^{-1}$ and $B$ have size respectively $n \times n$ and $n \times 1$, then $X$ is also well defined and it has size $n \times 1$. Thus the system has at least one solution, namely $X$.

Moreover, since $A$ is invertible, then we can use the matrix inverse algorithm to reduce the matrix $\begin{pmatrix} A & I_n \end{pmatrix}$ to $\begin{pmatrix} I_n & A^{-1} \end{pmatrix}$. Using the same sequence of row operations we can thus reduce

$$A \to I_n,$$

which implies that rank $(A) =$ rank $(I_n) = n$. Thus, from what we have seen in Lecture 5, the solution $X$ is unique. ■

**Example 9** Let us consider the system of linear equations

$$\begin{cases} 2x - 5y &= 1 \\ x + 2y &= 2 \end{cases}.$$

This system of linear equations corresponds to the matrix equation

$$AX = B \tag{2}$$

where

$$A = \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

From Example 2 we know that $A$ is invertible and that its inverse is

$$A^{-1} = \begin{pmatrix} \frac{2}{9} & \frac{5}{9} \\ -\frac{1}{9} & \frac{2}{9} \end{pmatrix}.$$

Thus. multiplying both sides of Equation 2 by $A^{-1}$ we obtain

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ IX &= \begin{pmatrix} \frac{2}{9} & \frac{5}{9} \\ -\frac{1}{9} & \frac{2}{9} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ X &= \begin{pmatrix} \frac{2}{9} + \frac{10}{9} \\ -\frac{1}{9} + \frac{4}{9} \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}. \end{aligned}$$

Hence, the solution of our system is $X = \begin{pmatrix} \frac{4}{3} & \frac{1}{3} \end{pmatrix}^T$.

**Example 10** Let us consider the system of 3 linear equations in 3 variables

$$\begin{cases} x + 2y - z &= 1 \\ 2x + 3y - 5z &= 2 \\ 4x + y + z &= -1 \end{cases}.$$

We can represent the system of linear equations using the matrix equation

$$AX = B$$

where

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -5 \\ 4 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

From Example 6 we know that $A$ is invertible and that its inverse is the matrix

$$A^{-1} = \frac{1}{26} \begin{pmatrix} -8 & 3 & 7 \\ 22 & -5 & -3 \\ 10 & -7 & 1 \end{pmatrix}$$

Then the unique solution of the system is

$$X = A^{-1}B = \frac{1}{26} \begin{pmatrix} -8 & 3 & 7 \\ 22 & -5 & -3 \\ 10 & -7 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{26} \begin{pmatrix} -9 \\ 15 \\ -5 \end{pmatrix},$$

that is, we have

$$x = -\frac{9}{26}, \quad y = \frac{15}{26} \quad \text{and} \quad z = -\frac{5}{26}.$$