Linear Algebra with Application
(LAWA 2021)

Lecture 8

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As in the previous lecture, let us consider matrices over $\mathbb{R}$.

1 Conditions for invertibility

The following result (given without proof) summarizes the relation between an invertible matrix and the associated system of linear equations.

Theorem 1 *Let A be a n-square matrix. The following conditions are equivalent:*

1. *The matrix A is invertible.*
2. *There exists a matrix C such that $AC = I$.*
3. *The matrix A can be carried to the identity matrix I by elementary row operations.*
4. *The system $AX = B$ has a solution for every choice of a column B.*

5. *The homogeneous system $AX = O$ has only the trivial solution $X_= = O$.*

Some of the equivalence in the previous theorem can be proved by using the definition of inverse and the results in Lectures 6 and 7.

We can also give an extra result, without proof, for invertible matrices.

Theorem 2 *Let A, C be two square matrices. If $AC = I$ then $CA = I$ also. Moreover, in this case, A and C are both invertible, $C = A^{-1}$ and $A = C^{-1}$*

Using the previous theorem we can show that the only invertible matrices are square matrices. That is, if A is an $m \times n$ matrix, and $AC = I_m$ and $CA = I_n$ hold for some $n \times m$ matrix C, then $m = n$.

This is false if A and C are not square matrices.

Example 3 Let us consider the two non-square matrices

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

One has

$$AC = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

but

$$CA = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \neq I_3.$$

Example 4 Let A, B be two square matrices and let us suppose that $A^3 = B$ and that B is invertible. Then, using Theorem 2, we can prove that A is invertible too (`Exercise`).

2 Elementary matrices

In Lecture 4 we defined the three types of elementary row operations on a matrix. Similary, we call *elementary column operations* on a matrix the following operations:

i) interchange two columns;

ii) multiply one of the columns by a nonzero number;

iii) add a multiple of one column to a different column.

Example 5 Let us consider the same matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

The three matrices

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 5 & 1 & 1 & 2 \\ 5 & -1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

are obtained from A using respectively an elementary row operation of type *i)* (interchanging column 2 with column 4), type *ii)* (multiplying the first column by 5), and type *iii)* (adding the second column to the third one).

A square matrix E obtained by doing a single elementary row operation <u>or</u> a single elementary column operation to the identity matrix I is called an *elementary matrix.*

We say that E is of type *i)*, *ii)* or *iii)* when the correspoding row or column operation is of type *i)*, *ii)* or *iii)*.

Example 6 The matrices

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

are elementary matrices of type *i)*, *ii)* and *iii)* respectively, obtained by performing the following row operations on the 3×3 identity matrix I_3:

$$I \xrightarrow[R_1 \leftrightarrow R_3]{i)} E_1, \quad I \xrightarrow[R_2 \to \frac{1}{3} R_2]{ii)} E_2 \quad \text{and} \quad I \xrightarrow[R_3 \to R_3 - 2R_1]{iii)} E_3.$$

Theorem 7 *Every elementary matrix E is invertible, and E^{-1} is the elementary matrix (of the same type of E) obtained from I by the inverse of the operation that produces E from I.*

Example 8 Let us consider the three elementary matrices E_1, E_2 and E_3 seen in Example 6. Then we can find their inverses E_1^{-1}, E_2^{-1} and E_3^{-1} (`Exercise`).

The left multiplication by an elementary matrix of a certain type is equivalent to a corresponding elementary row operation of the same type.

Example 9 Let us consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 3 & -2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Interchanging the 1-row and the 3-row (elementary row operation of type i)) can be performed by multiplying A by the matrix E_1 in Example 6. Indeed

$$E_1A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 3 & -2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

Subtracting 2 times the 1-row from the 3-row in the previous matrix (elementary row operation of type iii)) can be done by multiplying E_1A by the matrix E_3 in Example 6. Indeed

$$E_3(E_1A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Multiplying the 2-row by $\frac{1}{3}$ in the previous matrix can be done by left multiplication by the matrix E_2 in Example 6. Indeed

$$E_2(E_3E_1A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we can get an equivalent matrix to A in row echelon form by multiplying on the left by $(E_2E_3E_1)$. We can also obtain a reduced row-echelon form of A by left multiplying $E_2E_3E_1A$ by the elementary matrix of type iii)

$$E_4 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

indeed

$$E_4E_2E_3E_1A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, the right multiplication by an elementary matrix of a certain type is equivalent to a correspoding elementary column operation of the same type.

Example 10 Let us consider the matrix

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}.$$

Interchanging the first column with the third column (elementary column operation of type i) corresponds to multiply A on the right with the matrix E_1 of Example 6. Indeed

$$AE_1 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 \\ -1 & 2 & 0 \end{pmatrix}.$$

Subtracting 5 times the 1-column from the 3-column in the previous matrix (elementary operation of type *iii*)) can be performed by right multiplication by the elmentary matrix of type *iii*)

$$E_5 = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed

$$(AE_1)E_5 = \begin{pmatrix} 1 & 0 & 5 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 5 \end{pmatrix}.$$

Multiplying the 2-column of the previous matrix by $\frac{1}{2}$ is done by right multiplying AE_1E_5 by the elementary matrix of type *ii*)

$$E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed

$$(AE_1E_5)E_6 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 5 \end{pmatrix}.$$

By right multiplying $AE_1E_5E_6$ by the elmentary matrices of type *iii*)

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain the matrix

$$\begin{aligned} (AE_1E_5E_6)E_7E_8 &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} E_8 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

This matrix is called a *reduced column-echelon form* of A. Its transposition

$$(AE_1E_5E_6E_7E_8)^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is in reduced row-echelon form.

The previous examples can be generalized in the following result.

Theorem 11 *Let us consider two matrices A, B and let us suppose that there exists a series of row operations carrying $A \to B$. Then*

1. *There exists an invertible matrix U such that $B = UA$.*
2. *U can be constructed by performing the same row operations carrying A to B on the double matrix $\begin{pmatrix} A & I \end{pmatrix}$, that is*

$$\begin{pmatrix} A & I \end{pmatrix} \longrightarrow \begin{pmatrix} B & U \end{pmatrix}.$$

3. *$U = E_k \cdots E_2 E_1$, where $E_1, E_2, \ldots, E_k$ are the elementary matrices corresponding in order to the row operations carrying A to B.*

Example 12 Let us consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 3 & -2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

seen in Example 9. We have seen that this matrix can be carried to the matrix

$$B = \begin{pmatrix} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

by a series of elementary row operations, and that $B = UA$ where

$$U = E_4 E_2 E_3 E_1$$

is an invertible matrix with E_1, E_2, E_2 and E_4 elementary row matrices.

Example 13 Let us consider the matrix

$$A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & -1 & 0 \end{pmatrix}.$$

Using Theorem 11 we can find an invertible matrix U (with its decomposition in elementary matrices) and a matrix B in reduced row-echelon form such that $B = UA$ (`Exercise`).