Linear Algebra with Application
(LAWA 2021)

Lecture 9

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As in the previous lecture, let us consider matrices over $\mathbb{R}$.

1 Elementary matrices and rank

Combining Theorem 11 of Lecture 9 with Theorem 3 of Lecture 7 we obtain that the inverse of an invertible matrix can be written as a product of elementary matrices.

Example 1 Let us consider the invertible matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}.$$

A possible reduction of A is reduced row-echelon form is the following:

$$\begin{aligned}\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix} & \xrightarrow[R_1 \leftrightarrow R2]{i)} \begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}\\ & \xrightarrow[R_2 \to R_2 - 2R_1]{iii)} \begin{pmatrix}1 & -1\\ 0 & 3\end{pmatrix}\\ & \xrightarrow[R_2 \to \frac{1}{3}R_2]{ii)} \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\\ & \xrightarrow[R_1 \to R_1 + R2]{iii)} \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{aligned}$$

The elementary matrices corresponding to the previous elementary operations are, in order:

$$E_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad E_2 = \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}, \quad E_3 = \begin{pmatrix}1 & 0\\ 0 & \frac{1}{3}\end{pmatrix} \quad \text{and} \quad E_4 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

From Theorem 11 in Lecture 8 we thus have $I = A^{-1}A$, where

$$A^{-1} = E_4E_3E_2E_1 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & -2\end{pmatrix}.$$

Note that we can also write A as a product of elementary matrices. Indeed, since $A = \left(A^{-1}\right)^{-1}$, we have

$$\begin{aligned} A &= (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}\\ &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\end{aligned}.$$

If we combine row operations and column operations, we can get a simpler form of any matrix.

Theorem 2 *Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$ be a matrix of rank r. Then there exit two invertible matrices $U \in \mathcal{M}_{m,m}(\mathbb{R})$ and $V \in \mathcal{M}_{n,n}(\mathbb{R})$ such that*

$$UAV = \begin{pmatrix}I_r & O_{r,n-r}\\ O_{m-r,r} & O_{m-r,n-r}\end{pmatrix}$$

or, for short and when the size is clear from the context, $\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}$. *Moreover, the matrices U and V can be computed using the Gaussian Algorithm as follows:*

$$\begin{pmatrix}A & I_m\end{pmatrix} \to \begin{pmatrix}R & U\end{pmatrix},$$

where R is a reduced row-echelon matrix; and

$$\begin{pmatrix}R^T & I_n\end{pmatrix} \to \left(\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}^T \quad V^T\right).$$

Proof. [Idea] Let us give an idea of the proof.

First, we use a similar idea as the Matrix Inversion Algorithm. We add the $m \times m$ identity matrix to the right side of A to get the $m \times (m+n)$ matrix $\begin{pmatrix} A & I_m \end{pmatrix}$. Using the Gaussian algorithm we can perform a sequence of elementary row operation and obtain $\begin{pmatrix} R & U \end{pmatrix}$, where R is a reduced row-echelon matrix, equivalent to A, and U is the multiplication of the elementary matrices corresponding to the elementary row operations, according to Theorem 11 in Lecture 8.

If R is not already in the form $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, we consider its transpose R^T and we procede in a similar way. We add the $n \times n$ identity matrix to the right side of R^T, and doing a sequence of elementary row operations we obtain a matrix of the form $\left(\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}^T \quad V^T \right)$. From Theorem 11 in Lecture 8 it follows that V corresponds to the multiplication of the elementary matrices corresponding to the elementary column operations.

Since we have that $R = UA$ in the first step of the theorem, then we also have (always using Theorem 11 in Lecture 8)

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}^T = V^T R^T = V^T (UA)^T = V^T A^T U^T = (UAV)^T,$$

Recalling that for any matrix B we have $(B^T)^T = B$, we can conclude that

$$UAV = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}.$$

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Example 3 Let us consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ -1 & 2 & -1 & 1 \\ 2 & -4 & 5 & 1 \end{pmatrix}.$$

Let us use Theorem 2 to show that $\operatorname{rank}(A) = 2$ and that there exist two matrices U, V such that

$$UAV = \begin{pmatrix} I_2 & O \\ O & O \end{pmatrix}.$$

Let us first consider the reduction $\begin{pmatrix} A & I_3 \end{pmatrix} \to \begin{pmatrix} R & U \end{pmatrix}$ as in the first step of Theorem 2.

$$\begin{pmatrix} 1 & -2 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 1 & 0 & 1 & 0 \\ 2 & -4 & 5 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1}]{iii)} \begin{pmatrix} 1 & -2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \to \frac{1}{2} R_2]{ii)} \begin{pmatrix} 1 & -2 & 3 & 1 & & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -1 & & -2 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 + R_2}]{iii)} \begin{pmatrix} 1 & -2 & 0 & -2 & & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 1 & & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & & -\frac{3}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

Thus we have

$$R = \begin{pmatrix} 1 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

Note that here the reduced row-echelon matrix R has a unique form, while U may have different forms.

Moreover, since R has two leadings ones, we have $\operatorname{rank}(A) = \operatorname{rank}(R) = 2$.

Using the second step of Theorem 2, that is the reduction of $\begin{pmatrix} R^T & I_4 \end{pmatrix}$, we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 + 2R_1 \\ R_4 \to R_4 + 2R_1}]{iii)} \begin{pmatrix} 1 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{i)} \begin{pmatrix} 1 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_1]{iii)} \begin{pmatrix} 1 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & & 2 & 0 & -1 & 1 \end{pmatrix}.$$

$$= \left(\begin{pmatrix} I_2 & O_{2,1} \\ O_{2,2} & O_{2,1} \end{pmatrix} \quad V^T \right).$$

where

$$V = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the reduced row-echelon matrix $\begin{pmatrix} I_2 & O_{2,1} \\ O_{2,2} & O_{2,1} \end{pmatrix}$ equivalent to R has a unique form, while V may have different forms.

Finally, one can check that we actually have

$$\begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 & 1 \\ -1 & 2 & -1 & 1 \\ 2 & -4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

that is

$$UAV = \begin{pmatrix} I_2 & O_{2,2} \\ O_{1,2} & O_{1,2} \end{pmatrix}.$$

Example 4 Following the previous example, let us show that given the matrix

$$A = \begin{pmatrix} 3 & -3 & 6 \\ 1 & -1 & 1 \end{pmatrix}$$

we can write

$$UAV = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

for two invertible matrices U, V (`Exercise`).

2 Determinant

In Lecture 7 we defined the *determinant* of a generic 2×2-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as

$$\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

In this lecture we define the determinant of a generic square matrix and we show how to compute it.

To define the determinant, we give a recursive definition, that is we give a definition for a base case, here for a 1×1-matrix, and then we define the determinant of a $n \times n$-matrix using the determinant of a $(n-1) \times (n-1)$-matrix.

- Let $A = \begin{pmatrix} a \end{pmatrix} \in \mathcal{M}_{1,1}(\mathbb{R})$. Then $\det(A) = a$.
- Let $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in \mathcal{M}_{n,n}(\mathbb{R})$. Then

  $$\det(A) = a_{11}C_{11}(A) + a_{12}C_{12}(A) + \cdots + a_{1n}C_{1n}(A) \tag{1}$$

  where $C_{ij}(A)$ is called the (i,j)-*cofactor* of A and it is defined as

  $$C_{ij}(A) = (-1)^{i+j} \det(A_{ij}),$$

  for each i and j, where A_{ij} is the $(n-1) \times (n-1)$-matrix obtained from A by delating the i-row and the j-column. We also call $(-1)^{i+j}$ the *sign* of the (i,j)-position in A.

Equation (1) is called the *Laplace expansion*, or *cofactor expansion*, of A along the 1-row.

Example 5 The definition of determinant is consistent for 2×2-matrices. Indeed we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot (-1)^{1+1} \det (d) + b \cdot (-1)^{1+2} \det (c) = ad - bc.$$

Example 6 Let us find the determinant of a generic 3×3-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Using Equation (1) we have

$$\det A = a_{11}C_{11}(A) + a_{12}C_{12}(A) + a_{13}C_{13}(A).$$

From the definition of cofactor, it follows that

$$C_{11}(A) = (-1)^{1+1}\det (A_{11}) = (-1)^2 \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = a_{22}a_{33} - a_{23}a_{32},$$

$$C_{12}(A) = (-1)^{1+2}\det (A_{12}) = (-1)^3 \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} = -(a_{21}a_{33} - a_{23}a_{31}),$$

$$C_{13}(A) = (-1)^{1+3}\det (A_{13}) = (-1)^4 \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = a_{21}a_{32} - a_{22}a_{31}.$$

Thus, we have

$$\begin{aligned} \det A &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

Example 7 Following the previous example we can find the determinant of the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & 2 \\ 5 & 0 & 3 \end{pmatrix}$$

(`Exercise`).

When we have a lot of zeros, especially in the first row, the computation is easier.

Example 8 Let us compute the determinant of the matrix

$$A = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}.$$

By definition we have

$$\det(A) = a_{12}(-1)^{1+2}\det\begin{pmatrix} a_{21} & 0 & 0 \\ 0 & a_{33} & 0 \\ 0 & 0 & a_{44} \end{pmatrix}.$$

Let us now consider the 3×3-matrix

$$B = \begin{pmatrix} a_{21} & 0 & 0 \\ 0 & a_{33} & 0 \\ 0 & 0 & a_{44} \end{pmatrix}$$

Its determinant is given by

$$\det(B) = a_{21}(-1)^{1+1}\det\begin{pmatrix} a_{33} & 0 \\ 0 & a_{44} \end{pmatrix} = a_{21}\cdot 1\cdot(a_{33}a_{44}) = a_{21}a_{33}a_{44}.$$

Thus, we finally have

$$\det A = a_{12}\cdot(-1)\cdot(a_{21}a_{33}a_{44}) = -a_{12}a_{21}a_{33}a_{44}.$$

In the definition of determinant we considered the cofactor expansion along the first row. The following important result show us that we can compute it in a different way.

Theorem 9 (Laplace Expansion Theorem) *Let A be a square matrix. The determinant of A is equal to the cofactor expension along any row or along any column of A.*

Example 10 Let us consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 0 & 5 & 7 \\ 7 & 9 & 0 & 2 \\ 4 & 0 & 0 & 8 \end{pmatrix}.$$

A smart way to compute $\det(A)$ using a cofactor expansion along the rows and the column having the bigger number of zeros. Let us thus start by doing the cofactor expansion along the 4-row.

$$\begin{aligned} \det(A) &= 4\cdot(-1)^{4+1}\det\begin{pmatrix} -1 & 0 & 3 \\ 0 & 5 & 7 \\ 9 & 0 & 2 \end{pmatrix} + 0\cdot(-1)^{4+2}\det\begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 7 \\ 7 & 0 & 2 \end{pmatrix} \\ &\quad +0\cdot(-1)^{4+3}\det\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 7 \\ 7 & 9 & 2 \end{pmatrix} + 8\cdot(-1)^{4+4}\det\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 5 \\ 7 & 9 & 0 \end{pmatrix} \\ &= -4\det\begin{pmatrix} -1 & 0 & 3 \\ 0 & 5 & 7 \\ 9 & 0 & 2 \end{pmatrix} + 8\det\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 5 \\ 7 & 9 & 0 \end{pmatrix}. \end{aligned}$$

Applying the cofactor expansion along the 2-column of the first matrix and along the 3-column of the second matrix we obtain respectively:

$$\det \begin{pmatrix} -1 & 0 & 3 \\ 0 & 5 & 7 \\ 9 & 0 & 2 \end{pmatrix} = 5 \cdot (-1)^{2+2} \det \begin{pmatrix} -1 & 3 \\ 9 & 2 \end{pmatrix} = 5(-2-27) = -145$$

and

$$\det \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 5 \\ 7 & 9 & 0 \end{pmatrix} = 5 \cdot (-1)^{2+3} \det \begin{pmatrix} 2 & -1 \\ 7 & 9 \end{pmatrix} = -5(18+7) = -125.$$

Thus

$$\det(A) = -4 \cdot (-145) + 8 \cdot (-125) = -420.$$

The following results easily follows from Theorem 9

Corollary 11 *If a square matrix A has a row or a column of zeros, then $det(A) = 0$.*

Example 12 Let us consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Then, using the cofactor expansion along the second row, we find

$$\det(A) = -0 \cdot \det \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = 0.$$