

# MPI - Lecture 3

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Outline

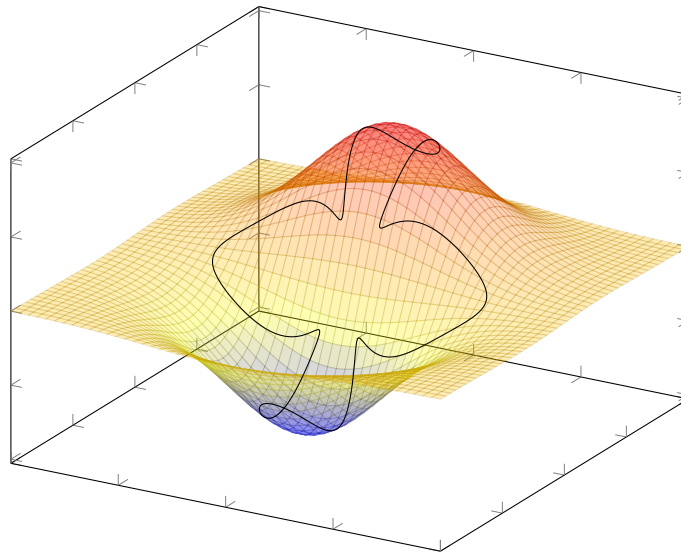
- Constrained optimization
- Reminder: integration of functions of 1 variable
- 2-variate function integration

## Constrained optimization

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Motivation

Find the maximum and minimum points when walking along the black line:



The problem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

Find (local) maxima and minima of  $f$  subject to

$$\begin{cases} g_1(x_1, x_2, \dots, x_n) = 0 \\ g_2(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ g_p(x_1, x_2, \dots, x_n) = 0. \end{cases}$$

Set  $\mathcal{G} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid g_i(x_1, x_2, \dots, x_n) = 0, i = 1, 2, \dots, p\}$ .

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Assumptions

1. The functions  $f$  and  $g_i$ , with  $i = 1, 2, \dots, p$ , have continuous second partial derivatives.
2. The gradients  $\nabla g_1(x), \nabla g_2(x), \dots, \nabla g_p(x)$  form a linearly independent set for all  $x \in \mathcal{G}$ .

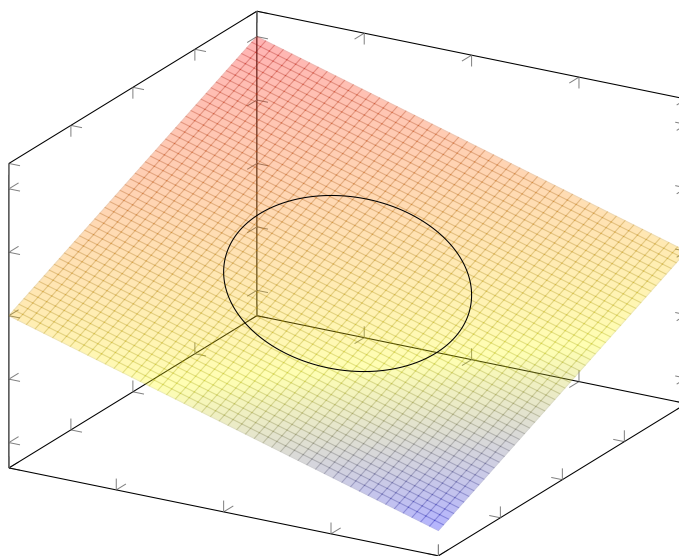
**Example 1.** Are the gradients of the following functions linearly independent?

$$\begin{aligned} g_1(x, y) &= 2x + xy^2, & g_2(x, y) &= 4x + 2xy^2, \\ g_3(x, y) &= 2xy^2 + 4y^2, & g_4(x, y) &= 2x + 3xy^2 + 4y^2. \end{aligned}$$

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$$f(x, y) = x + y, \quad \mathcal{G} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2\}$$

Running example




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Necessary condition

**Theorem 2.** Assume  $f$  has a local extremum in  $x^* = (x_1^*, \dots, x_n^*) \in \mathcal{G}$  subject to  $\mathcal{G}$ .

Then there exist numbers  $\mu_1^*, \dots, \mu_p^*$  such that the *Lagrangian function*  $L$  given by

$$L(x_1, \dots, x_n, \mu_1, \dots, \mu_p) = f(x_1, \dots, x_n) + \sum_{i=1}^p \mu_i g_i(x_1, \dots, x_n)$$

has zero partial derivatives with respect to  $x_1, \dots, x_n$  at the point  $x^*$ .

In other words, the following system is true:

$$\begin{cases} \frac{\partial f}{\partial x_1}(x^*) + \mu_1^* \frac{\partial g_1}{\partial x_1}(x^*) + \dots + \mu_p^* \frac{\partial g_p}{\partial x_1}(x^*) = 0 \\ \vdots \\ \frac{\partial f}{\partial x_n}(x^*) + \mu_1^* \frac{\partial g_1}{\partial x_n}(x^*) + \dots + \mu_p^* \frac{\partial g_p}{\partial x_n}(x^*) = 0 \end{cases}$$

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Sufficient condition

**Theorem 3.** Let  $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$  and  $\mu^* = (\mu_1^*, \dots, \mu_p^*) \in \mathbb{R}^p$  such that

- (i) the Lagrangian function  $L(x_1, \dots, x_n, \mu_1, \dots, \mu_p)$  has zero partial derivatives with respect to  $x_1, \dots, x_n$  at the point  $(x^*, \mu^*) \in \mathbb{R}^{n+p}$ ;
- (ii) the Lagrangian function  $L(x_1, \dots, x_n, \mu_1, \dots, \mu_p)$  has zero partial derivatives with respect to  $\mu_1, \dots, \mu_p$  at the point  $(x^*, \mu^*) \in \mathbb{R}^{n+p}$ ;
- (iii) for all non-zero  $y \in \mathbb{R}^n$  satisfying  $y \cdot \nabla g_i(x^*) = 0$  for  $i = 1, 2, \dots, p$  we have

$$y \left( \nabla^2 f(x^*) + \sum_{i=1}^p \mu_i^* \nabla^2 g_i(x^*) \right) y^T > 0.$$

Thus, the function  $f$  has a strict local minimum at  $x^*$  (subject to  $\mathcal{G}$ ).

If we replace in (iii) the condition “ $> 0$ ” by “ $< 0$ ”, we obtain a sufficient condition of a strict local maximum.

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Example

**Example 4.** Find maxima and minima of  $f(x, y) = x + y$  subject to  $x^2 + y^2 = 2$ .

## Integration of functions of 1 variable

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a < b$ .

Recall what does  $\int_a^b f(x) dx$  mean, if it exists.

What is its geometrical meaning?

Integration of  
functions of 1  
variable

Definition

Let  $\Delta = (x_i)_{i=0}^n$  define a partition of  $[a, b] : a = x_0 < x_1 < \dots < x_n = b$ .

Set  $F_{\Delta,i} = \max_{x \in [x_{i-1}, x_i]} f(x)$  and  $f_{\Delta,i} = \min_{x \in [x_{i-1}, x_i]} f(x)$ .

The upper Darboux sum of  $f$  with respect to the partition  $\Delta$  is

$$S_{f,\Delta} = \sum_{i=1}^n F_{\Delta,i} \cdot (x_i - x_{i-1})$$

and the lower Darboux sum of  $f$  with respect to the partition  $\Delta$  is

$$s_{f,\Delta} = \sum_{i=1}^n f_{\Delta,i} \cdot (x_i - x_{i-1}).$$

Definition...

The upper Darboux integral (of  $f$  over  $[a, b]$ ) is

$$D_f = \inf\{S_{f,\Delta} : \Delta \text{ is a partition of } [a, b]\}$$

and the lower Darboux integral (of  $f$  over  $[a, b]$ ) is

$$d_f = \sup\{s_{f,\Delta} : \Delta \text{ is a partition of } [a, b]\}.$$

If  $D_f = d_f$ , we call this value the Darboux integral of  $f$  over  $[a, b]$ , and denote it

$$\int_a^b f(x) dx = D_f = d_f.$$

We say that  $f$  is (Darboux-)integrable over  $[a, b]$ .

This is equivalent to the Riemann integral and to Riemann integrability.

A few properties

If  $f$  is continuous on  $[a, b]$ , then it is integrable on  $[a, b]$ .

Let  $f$  be integrable on  $[a, b]$  and on  $[b, c]$  (with  $a < b < c$ ).

We have that  $f$  is integrable on  $[a, c]$  and

$$\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.$$

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Primitive function

Let  $F(x)$  be a real function which is continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

Let  $f(x)$  be a real function which is continuous on  $(a, b)$  and such that

$$\forall x \in (a, b), \quad F'(x) = f(x).$$

Such function  $F$  is called a primitive function of  $f$  on  $(a, b)$ .

**Example 5.** Find a primitive function on  $(0, 1)$  of the function  $f(x) = 2x + x^2$ .

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Newton's formula

Let  $f$  be integrable on  $[a, b]$  and  $F(x)$  be (one of) its primitive function on  $(a, b)$ .

We have

$$\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a).$$

**Example 6.** Calculate  $\int_0^1 (2x + x^2) \, dx$ .

Let  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha < \beta$ .

Let  $\varphi$  be a real function differentiable on  $(\alpha, \beta)$  such that  $\varphi$  and  $\varphi'$  are both continuous on  $[\alpha, \beta]$ .

Let  $f$  be continuous on  $[\varphi(\alpha), \varphi(\beta)]$  (or if  $\varphi(\alpha) \leq \varphi(\beta)$ , otherwise continuous on  $[\varphi(\beta), \varphi(\alpha)]$ ).

If  $f(\varphi(t)) \varphi'(t)$  is integrable on  $[\alpha, \beta]$ , then

$$\int_{\alpha}^{\beta} f(\varphi(t)) \varphi'(t) dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) dx.$$

**Example 7.** Calculate  $\int_1^2 \frac{2 \ln(t)^2}{t} dt$ .

Let  $f$  and  $g$  be differentiable on  $(a, b)$  and let  $f, g, f', g'$  be continuous on  $[a, b]$ .

We have

$$\int_a^b f'(x)g(x) dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x) dx.$$

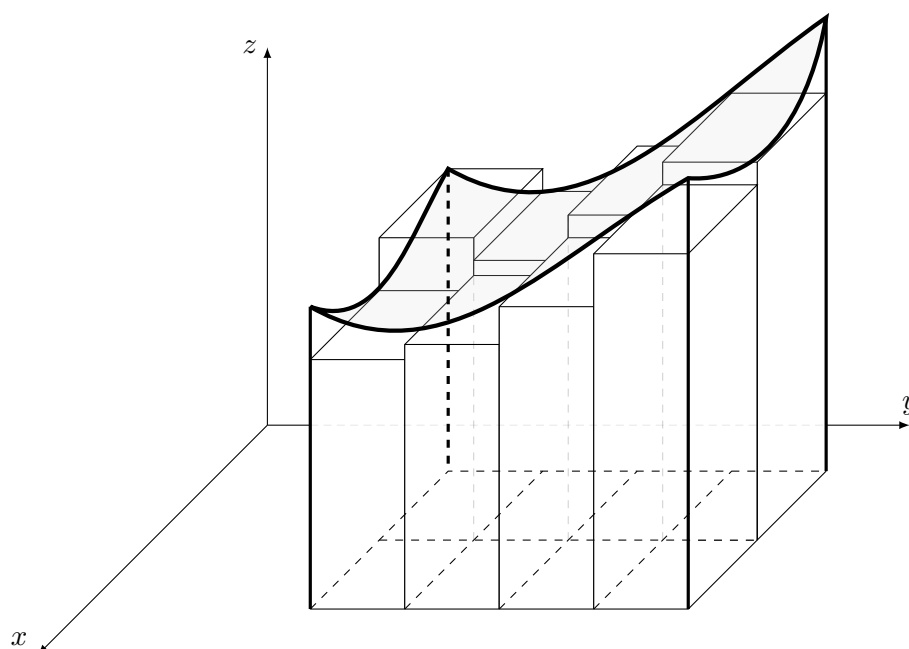
**Example 8.** Calculate  $\int_1^2 10x \ln x dx$ .

## 2-variate function integration

2-variate  
function

Suppose we have a function  $f : D \rightarrow \mathbb{R}$ , where  $D = [a, b] \times [c, d]$ .

Imagine that this function represents (part of) a surface of some object.  
What is the volume of this object?



Definition

Let  $\Delta_x = (x_i)_{i=0}^n$  define a partition of  $[a, b]$  and  $\Delta_y = (y_j)_{j=0}^m$  a partition of  $[c, d]$ .

Then,  $\Delta = \Delta_x \times \Delta_y$  defines a partitions of  $D = [a, b] \times [c, d]$  into rectangles.

Set

- $F_{\Delta, i, j} = \max \{f(x, y) : (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$  and
- $f_{\Delta, i, j} = \min \{f(x, y) : (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$ .



The upper Darboux sum of  $f$  with respect to the partition  $\Delta$  is

$$S_{f,\Delta} = \sum_{i=1}^n \sum_{j=1}^m F_{\Delta,i,j} \cdot (x_i - x_{i-1}) \cdot (y_j - y_{j-1})$$

while the lower Darboux sum of  $f$  with respect to the partition  $\Delta$  is

$$s_{f,\Delta} = \sum_{i=1}^n \sum_{j=1}^m f_{\Delta,i,j} \cdot (x_i - x_{i-1}) \cdot (y_j - y_{j-1}).$$

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Definition...

The upper Darboux integral (of  $f$  over  $D$ ) is

$$D_f = \inf \{S_{f,\Delta} : \Delta \text{ is a (rectangular) partition of } D\}$$

and the lower Darboux integral (of  $f$  over  $D$ ) is

$$d_f = \sup \{s_{f,\Delta} : \Delta \text{ is a (rectangular) partition of } D\}.$$

If  $D_f = d_f$ , we call this value the (double) Darboux integral of  $f$  over  $D$ , and denote it

$$\iint_D f(x, y) \, dx \, dy = D_f = d_f.$$

We say that  $f$  is (Darboux-)integrable over  $D$ .

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How to calculate a double integral?

The following statement can be derived from the definition.

**Theorem 9.** *If  $f$  is integrable over  $D = [a, b] \times [c, d]$  and one of the integrals*

$$\int_a^b \left( \int_c^d f(x, y) \, dy \right) dx \quad \text{or} \quad \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy$$

*exists, then it is equal to*

$$\iint_D f(x, y) \, dx \, dy.$$

**Example 10.** Calculate the double integral over  $D = [0, 2] \times [-1, 2]$  of the function  $f(x, y) = x^2y + 1$ .

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And if  $D$  is not a rectangle?

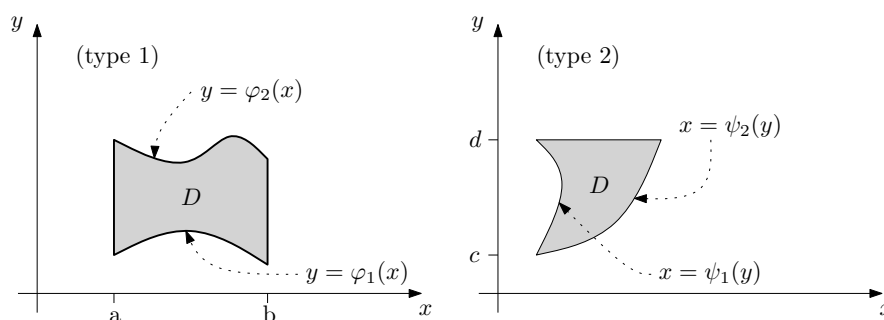
The definition is very similar: we approximate  $D$  using smaller and smaller rectangular areas...

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Special types of domain  $D$  (1/2)

We will consider the following two types of the domain  $D$ .

- (type 1)  $x \in [a, b]$  and  $y$  is bounded by continuous functions  $\varphi_1(x)$  and  $\varphi_2(x)$ ;
- (type 2)  $y \in [c, d]$  and  $x$  is bounded by continuous functions  $\psi_1(y)$  and  $\psi_2(y)$ .




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Special types of domain  $D$  (2/2)

Double integrals over such  $D$  are calculated as follows.

**Theorem 11.** If the integral on the right side exists, then we have (for such a domain  $D$ ):

- if  $D$  is of type 1, then

$$\iint_D f(x, y) dx dy = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \right) dx;$$

- if  $D$  is of *type 2*, then

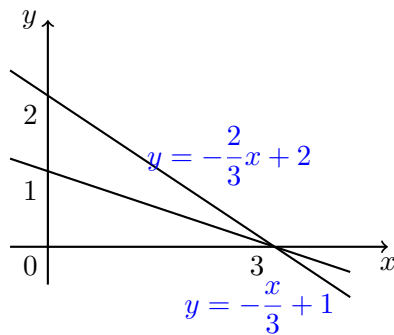
$$\iint_D f(x, y) dx dy = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \right) dy.$$

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Example

**Example 12.** Let  $D$  be the region given by the triangle with vertices  $(0, 1)$ ,  $(0, 2)$  and  $(3, 0)$ . Calculate

$$\iint_D \frac{x+y}{2} dx dy.$$



$$\int_0^3 \int_{1-\frac{x}{3}}^{2-\frac{2}{3}x} \frac{x+y}{2} dy dx = \dots = \frac{3}{2}$$