MPI - Lecture 4

Outline

- Introduction and motivation
- Hierarchy of sets with one binary operation
  - Introduction
  - Definitions and elementary properties
  - Cayley table
  - Cayley graph

Introduction and motivation

Searching for hidden similarities. . .

Let us consider this objects:

- the set $\mathbb{Z}$ of integers with the usual sum;
- the set of matrices $\mathbb{R}^{n,n}$ with the operation of matrix multiplication;
- the set of relations on a set A with the operation of relation composition;
- the set $\{0, 1, 2, 3\}$ with the multiplication $\pmod 4$;
- the set of finite automata with the operation of composition;
- the set of all colors with the operation "mixing";
- . . .

What do they have in common?

Still the same structure!

All presented objects have the same structure. Indeed, they consist of two ingredients:

- A (finite or infinite) **set of objects**.
- A **binary operation** mapping two objects onto (exactly) one object (from the same set of objects).

Generally, we speak about a pair of: a **set** and a **binary operation** on it.

We will (mostly) use one of the following notations: $(M, \cdot)$ (multiplicative notation), $(M, +)$ (additive notation), or $(M, \circ)$ (general notation), where

- $M \neq \emptyset$ is a set, and
- for binary operation we have $\cdot : M \times M \to M$ (resp. $+ : M \times M \to M$, resp. $\circ : M \times M \to M$).

What is going on in algebra?

The pair of "a set and a binary operation on it" could represent very different structures. We shall classify them by their properties.

We are interested in properties of the binary operation:

1. Is it associative?
2. It is commutative?
3. Are there some neutral elements for the binary operation?

Why are we doing this?

If we prove some statement for a general structure $(M, \cdot)$, where $\cdot$ is an associative operation, this statement is proved for all particular structures with an associative binary operation! A proof of this statement is reduced to a proof of associativity of the operation! We can understand a general structure as a **parent object**, from which particular structures **inherit** all its properties (see below).

Example of "inheritance" (1/4)

On the set of non-zero real numbers we prove the following (trivial) theorem:

Theorem 1. *For all $b, c \in \mathbb{R} \setminus \{0\}$, the equation $bx = c$ has solution $x = b^{-1}c$.*

Proof.

$$\begin{aligned} bx &= c && \text{[multiplication on the left by the inverse element } b^{-1}\text{]} \\ b^{-1}(bx) &= b^{-1}c && \text{[moving brackets due to associativity]} \\ (b^{-1}b)x &= b^{-1}c && \text{[for arbitrary } b \text{ we have } b^{-1}b = 1\text{]} \\ 1x &= b^{-1}c && \text{[for arbitrary } x \text{ we have } 1x = x\text{]} \\ x &= b^{-1}c && \end{aligned}$$

□

What was fundamental for the proof: associativity, existence of (left) inverse element, existence of the neutral element.

Example of "inheritance" (2/4)

Let us consider a set M of all matrices $\mathbb{R}^{n,n}$ with the operation of matrix multiplication.

- Is the matrix multiplication associative? Yes. For $\forall\, A, B, C \in M$ we have $A(BC) = (AB)C$.
- Is there a neutral element? Yes. The identity matrix I_n has the property $I_n A = A$ valid for all $A \in M$.
- Is there an inverse matrix for all $A \in M$? No! We have to restrict ourselves to the set of regular matrices M_{reg}.

Example of "inheritance" (3/4)

We have everything needed to prove the theorem for matrices.

Theorem 2. *For all $B, C \in M_{reg}$, the equation $BX = C$ has solution $X = B^{-1}C$.*

Proof.

$$\begin{aligned} BX &= C && \text{[multiplication on the left by the inverse element } B^{-1}\text{]} \\ B^{-1}(BX) &= B^{-1}C && \text{[moving brackets due to associativity]} \\ (B^{-1}B)X &= B^{-1}C && \text{[for arbitrary } B \text{ we have } B^{-1}B = I_n\text{]} \\ I_nX &= B^{-1}C && \text{[for arbitrary } C \text{ we have } I_nX = X\text{]} \\ X &= B^{-1}C && \end{aligned}$$

□

What was fundamental for the proof: associativity, existence of (left) inverse element, existence of the neutral element.

Example of "inheritance" (4/4)

Suppose that we are given a pair $(M, \cdot)$ where the associativity law holds, for each element $b \in M$ there exists an inverse element, denoted by b^{-1}, and there exists a neutral element e. We will call such pair a group.

We have a general theorem.

Theorem 3. *For arbitrary elements b, c of a group $(M, \cdot)$, the equation $bx = c$ has solution $x = b^{-1}c$.*

Proof.

$$\begin{array}{rcll} bx & = & c & \text{[multiplication on the left by the inverse element } b^{-1}\text{]} \\ b^{-1}(bx) & = & b^{-1}c & \text{[moving brackets due to associativity]} \\ (b^{-1}b)x & = & b^{-1}c & \text{[for arbitrary } b \text{ we have } b^{-1}b = e\text{]} \\ ex & = & b^{-1}c & \text{[for arbitrary } x \text{ we have } 1x = x\text{]} \\ x & = & b^{-1}c & \end{array}$$

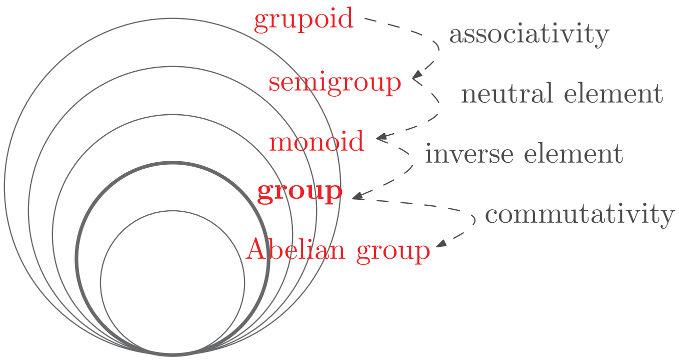
□

Hierarchy of sets with one binary operation

Introduction

Sets with one binary operation

We call an arbitrary pair "a set and a binary operation" a groupoid. Adding another requirements we get further notions.



Examples

- For the pair $(\mathbb{R} \setminus \{0\}, \cdot)$, the associative and commutative laws hold, the neutral element is 1 and the inverse element for b is $b^{-1} = 1/b$.

  It is an Abelian group.

- For the pair $(\mathbb{Z}, +)$ associative and commutative laws hold, the neutral element is 0 and the inverse element for b is $b^{-1} = -b$.

  It is an Abelian group.

- For the pair $(M_{\text{reg}}, \cdot)$ associativity law holds, the neutral element and the inverse exist, but the commutative law is not valid!

  It is a group, but not Abelian.

Mathematical analogy to Object-oriented programming

We can consider the groupoid, monoid, etc., as mathematical (abstract) objects, for which a nonempty set and a binary operation with given properties are defined.

For this abstract classes we can prove various statements (for example the theorem on solving linear equation for groups).

If for some particular pair $(M, \circ)$ we prove that it is a groupoid, monoid, etc., it means that it "inherits" all this statements and we don't need to prove them separately!

This analogy could be employed in real programming.

Definitions and elementary properties

Groupoid, semigroup, monoid, group

Definition 4.
- *An ordered pair $(M, \circ)$, where M is an arbitrary nonempty set and $\circ$ is a binary operation on M, is called a groupoid.*
- *A groupoid $(M, \circ)$ such that $\circ$ is associative is called a semigroup.*
- *A semigroup $(M, \circ)$ such that there exists a neutral element e satisfying*

  $$\forall\, a \in M \quad \textit{holds} \quad e \circ a = a \circ e = a$$

  is called a monoid.
- *A monoid $(M, \circ)$ such that for each $a \in M$ there exists an inverse element $a^{-1} \in M$ satisfying*

  $$a^{-1} \circ a = a \circ a^{-1} = e$$

  is called a group.
- *Moreover, if $\circ$ is commutative, we say that a group $(M, \circ)$ is a commutative (or Abelian) group.*

Set closed under the binary operation. What does it mean?

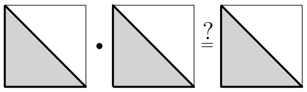
In the definition we require the binary operation $\circ$ to be a "binary operation on M".

This means that the result of a binary operation applied on two elements from M again belongs to M – we say that the **set M is closed under $\circ$.**

Example 5. *The pair $(\mathbb{Z}_-, \cdot)$ of negative integers with the usual multiplication is not even a groupoid, because it is not closed under the operation: $(-1) \cdot (-1) = 1 \notin \mathbb{Z}_-$.*

Whether the set is or is not closed under the binary operation is not always obvious.

Example 6. *Let us consider the couple* $(M_{triang}, \cdot)$ *of lower triangular matrixes with the usual matrix multiplication. Is* M_{triang} *closed under the operation* $\cdot$*?*



Manual for classification of sets with binary operation

If we have a given pair “a set and a binary operation” and we want to find out whether it is a groupoid, semigroup, monoid, (Abelian) group, we can proceed this way:

1. Is the set closed under the operation? If `yes`, it is a groupoid; if `not`, END.
2. Does the associativity law hold? If `yes`, it is a semigroup; if `not`, END.
3. Is there a neutral element? If `yes`, it is a monoid; if `not`, END.
4. Is there an inverse to each element? If `yes`, it is a group; if `not`, END.
5. Does the commutativity law hold? If `yes`, it is an Abelian group; if `not`, END.

Mostly “proofs” in these individual steps are very easy or obvious. Sometimes, they only *seem* obvious.

Groupoid, semigroup, monoid, group – examples (1/4)

Example 7. *Let us consider the groupoid* $(\mathbb{Q}, \circ)$*, where the binary operation* $\circ$ *is defined as the arithmetic mean:*

$$a \circ b = \frac{a+b}{2}.$$

Is this structure a semigroup / monoid / group?

In a semigroup, the associative law must hold. Let us claim that for the operation $\circ$ *the law* <u>*does not hold*</u>*, and let us prove it by a* <u>*counterexample*</u>*:*

$$(2 \circ -2) \circ 4 = 0 \circ 4 = 2 \quad \textit{but} \quad 2 \circ (-2 \circ 4) = 2 \circ 1 = \frac{3}{2}.$$

So, the associative law does not hold, and the structure is not a semigroup. It follows that $\mathbb{Q}$ *with this operation is neither a monoid nor a group.*

Groupoid, semigroup, monoid, group – examples (2/4)

Example 8. *Let us consider a groupoid* $(\mathbb{R}^+, \circ)$, *where the binary operation* $\circ$ *is defined as follows:*

$$a \circ b = \frac{a \cdot b}{a + b}.$$

- *Is* $(\mathbb{R}^+, \circ)$ *a semigroup?*
- *Is* $(\mathbb{R}^+, \circ)$ *a monoid?*

Groupoid, semigroup, monoid, group – examples (3/4)

Example 9. *Let us consider a groupoid* $(\mathbb{R}, \cdot)$, *where the binary operation is the usual multiplication of numbers.*

- *Is it a semigroup?*
- *Is it a monoid?*
- *Is it a group?*

Groupoid, semigroup, monoid, group – examples (4/4)

From the definition it follows that each group is a monoid, each monoid is a semigroup and each semigroup is a groupoid. Written in symbols we get:

$$\text{groupoid} \supset \text{semigroup} \supset \text{monoid} \supset \text{group} .$$

From the previous three examples we can be even more specific:

$$\text{groupoid} \supsetneq \text{semigroup} \supsetneq \text{monoid} \supsetneq \text{group} ,$$

because we have found a groupoid that is not a semigroup, a semigroup that is not a monoid, and a monoid that is not a group.

Uniqueness of neutral element

Theorem 10. *Given a monoid, there exists exactly one neutral element.*

Proof. Let $(M, \circ)$ be a monoid and e some neutral element (by definition we know that at least one exists!).

We prove by contradiction that e is the only neutral element.

By contradiction, assume that in the monoid there exists another neutral element $\overline{e}$ different from e.

Using the property of the neutral element, it holds that

$$\overline{e} = \overline{e} \circ e = e.$$

We get a contradiction with the assumption that $\overline{e} \neq e$. □

Uniqueness of the inverse element

Theorem 11. *Given a group, each element has exactly one inverse element.*

Proof. Let $(G, \circ)$ be a group, a an arbitrary element of the group and a^{-1} one of its inverse elements (from the definition of a group we know that there exists at least one!).

We prove *by contradiction* that a^{-1} is the only one.

Assume that there exists another inverse element $\overline{a}$ different from a^{-1}. Hence it holds that

$$\overline{a} = \overline{a} \circ e = \overline{a} \circ \left(a \circ a^{-1}\right) = (\overline{a} \circ a) \circ a^{-1} = e \circ a^{-1} = a^{-1}$$

where e is the unique neutral element.

Thus we get a contradiction with the assumption that $\overline{a} \neq a^{-1}$. □

Cayley table

Cayley tables for finite groups

If the set M from the pair $(M, \circ)$ has a finite number of elements, its structure (with the given operation $\circ$) could be completely represented by the Cayley table.

Its construction is obvious from the following example.

Example 12. *Let us consider* $(\mathbb{Z}_4, +_4)$, *i.e., the set of numbers* $\{0, 1, 2, 3\}$ *with addition modulo* 4. *Since the set has* 4 *elements, the Cayley table has* 4 *rows and* 4 *columns:*

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

So, in the cell in row m and column n we write the result of $m +_4 n = m + n \ (mod\ 4)$.

For example the cell in row 2 and column 3 is filled with $2+3 \ (mod\ 4) = 1$.

What can be easily read from a Cayley table

Cayley table offers all information about a given set and operation.

Some properties are very easy to read from the table; others with some difficulty:

- The set M is closed under the operation $\circ$ if all cells of the table contain elements from the set M only.
- The associativity law is difficult to read.
- The neutral element e is the one for which the corresponding row and column are just a copy of the first row and the first column of the table.
- The inverse element to the element a is the one corresponding to the row and column where the neutral element e is placed.
- ...

Cayley table and latin square (1/4)

Question: Is it possible to recognize whether a table is a Cayley table of a group? **Answer**: Almost.

Theorem 13. *The Cayley table of each group forms a latin square.*

A latin square for a set M of n elements is a matrix $n \times n$ such that each row and column contains all elements of the set M.

We prove the theorem by proving another one from which the statement of the original theorem follows directly.

Unfortunately, not each Cayley table forming a latin square is a Cayley table of a group. Later we present a counterexample.

Cayley table and latin square (2/4)

Theorem 14. *In each group, we can divide uniquely. In other words: in each group $(G, \circ)$, for arbitrary $a, b \in G$ the equations*

$$a \circ x = b \quad \text{and} \quad y \circ a = b$$

have only one solution.

Proof. Since we are in a group, each element has only one inverse.

The only solutions of the equations are $x = a^{-1} \circ b$ and $y = b \circ a^{-1}$. □

It is possible to prove that a group is a semigroup with a "unique division", i.e., the unique division guarantees the existence of a neutral element and inverse.

Cayley table and latin square (3/4)

Now we prove the theorem saying that the Cayley table of group is a latin square.

Proof. Proof by contradiction.

Let us suppose that the table of some group $(G, \circ)$ is not a latin square.

Hence, in some row or column there is one element, denote it as b, repeated twice. WLOG[1], assume that it happens in row n and columns m_1 and m_2.

$\circ$	$\cdots$	m_1	$\cdots$	m_2	$\cdots$
$\vdots$		$\vdots$		$\vdots$	
n	$\cdots$	b	$\cdots$	b	$\cdots$
$\vdots$		$\vdots$		$\vdots$	

It follows that the equation $n \circ x = b$ has two different solutions, namely m_1 and m_2, which is a **contradiction with the previous theorem**! □

Cayley table and latin square (4/4)

We have shown that the fact that a Cayley table is a latin square is a *necessary* condition for the given set and operation to be a group.

The following example says it is not a *sufficient* condition.

[1] Without Loss Of Generality

Example 15. *Let us consider a set $M = \{a, b, c\}$ with operation given by the Cayley table:*

$\circ$	a	b	c
a	b	a	c
b	c	b	a
c	a	c	b

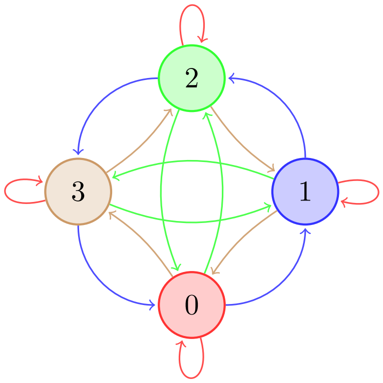
This table creates a latin square; in spite of it, it is not the table of a group (Why?!).

Cayley graph

Cayley graph of a group

A finite Abelian group $G = (M, \circ)$ may be visualised by a Cayley graph with

- set of vertices V being the elements of G, i.e., $V = M$,
- set of directed edges E the set of (ordered) pairs (a, b) such that $b = a \circ c$ for some $c \in M$ (or, as we can see, for some $c \in N$ with N a subset of M).



If the group in question is not Abelian, we need to depict edges (a, b) for $b = c \circ a$ for some $c \in M$.