MPI - Lecture 6

Outline

- Homomorphisms
- Application of groups theory in cryptography

Homomorphisms

Motivation

The same groups and distinct elements (1/5)

$\mathbb{Z}_5^\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

order: 4 [2mm] subgroups: $\{1\}$, $\{1,4\}$, $\{1,2,3,4\}$ [2mm] neutral element: 1 [2mm] inverse elements: $1^{-1}=1,\quad 2^{-1}=3,$ $3^{-1}=2,\quad 4^{-1}=4.$

$\mathbb{Z}_4^+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

order: 4 [2mm] subgroups: $\{0\}$, $\{0,2\}$, $\{0,1,2,3\}$ [2mm] neutral element: 0 [2mm] inverse elements: $-0=0,\quad -1=3,$ $-2=2,\quad -3=1.$ Aren't these two groups in fact the same group differing only in the "names" of their elements?

The same groups and distinct elements (2/5)

$\mathbb{Z}_5^\times$	10	23	31	42
10	10	23	31	42
23	23	42	10	31
31	31	10	42	23
42	42	31	23	10

$\mathbb{Z}_4^+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\mathbb{Z}_4^+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Let us try to rename the elements of the group $\mathbb{Z}_5^\times$ so to get $\mathbb{Z}_4^+$:

- The neutral element has very special and unique properties: we rename 1 to 0.
- If the complete structure should be preserved, then the only two-elements subgroup $\{1, 4\}$ (in $\mathbb{Z}_5^\times$) must correspond to the subgroup $\{0, 2\}$ (in $\mathbb{Z}_4^+$): we map $4 \leftrightarrow 2$.
- Now, it remains to rename only 2 and 3; we can check that both remaining possibilities work; we choose, for instance, $3 \leftrightarrow 1$ and $2 \leftrightarrow 3$.
- It suffices to reorder the rows…and we have the Cayley table of $\mathbb{Z}_4^+$.

The same groups and distinct elements (3/5)

We have found a way to rename the elements in one table to gain an exact copy of the other table (after rearranging rows and columns).

This renaming is actually an **injective** mapping of the set $\{1, 2, 3, 4\}$ **onto** the set $\{0, 1, 2, 3\}$; let us denote it φ_1:

$$\varphi_1(1) = 0, \qquad \varphi_1(2) = 3, \qquad \varphi_1(3) = 1, \qquad \varphi_1(4) = 2.$$

We have pointed out that the mapping φ_2 works as well:

$$\varphi_2(1) = 0, \qquad \varphi_2(2) = 1, \qquad \varphi_2(3) = 3, \qquad \varphi_2(4) = 2.$$

Would all bijections do the same job? And if not, what makes these two so special?

The same groups and distinct elements (4/5)

Let us rename the elements of the group $\mathbb{Z}_5^\times$ according to the bijection φ_3:

$$\varphi_3(1) = 0, \qquad \varphi_3(2) = 3, \qquad \varphi_3(3) = 2, \qquad \varphi_3(4) = 1.$$

$\mathbb{Z}_5^\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\varphi_3(\mathbb{Z}_5^\times)$	0	3	2	1
0	0	3	2	1
3	3	1	0	2
2	2	0	1	3
1	1	2	3	0

$\mathbb{Z}_4^+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

The resulting table is not the Cayley table of the group $\mathbb{Z}_4^+$, because, e.g., $3 + 3 \pmod 4 \neq 1$.

The bijection φ_3 does not give rise to the same structure of the group $\mathbb{Z}_4^+$; only φ_1 and φ_2 have this property.

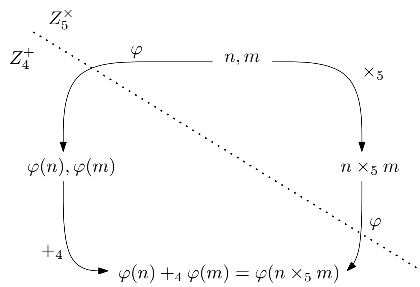
The same groups and distinct elements (5/5)

The desired property, which only the bijections φ_1 and φ_2 have, is the following:

$$\text{for all } n, m \in \{1, 2, 3, 4\}, \text{ we have } \varphi(n \times_5 m) = \varphi(n) +_4 \varphi(m),$$

where $\times_5$ denotes the operation in the group $\mathbb{Z}_5^\times$, and $+_4$ the one in the group $\mathbb{Z}_4^+$.

In words: If we apply the operation $\times_5$ to two arbitrary elements of the group $\mathbb{Z}_5^\times$ and then we send the result to $\mathbb{Z}_4^+$ by φ, we obtain the same result as when we first transform by φ the elements to $\mathbb{Z}_4^+$ and **then** apply the operation $+_4$.



Definition and properties

Homomorphism and isomorphism

Definition 1. *Let* $G = (M, \circ_G)$ *and* $H = (N, \circ_H)$ *be two groupoids. The mapping* $\varphi : M \to N$ *is a homomorphism from* G *to* H *if*

$$\text{for all } x, y \in M, \text{ we have } \quad \varphi(x \circ_G y) = \varphi(x) \circ_H \varphi(y).$$

If, moreover, φ *is injective (resp. surjective, resp. bijective) we say that* φ *is a monomorphism (resp. epimorphism, resp. isomorphism).*

A homomorphism preserves the structure given by the binary operation: the result is the same if we first apply the operation and then the homomorphism or if we proceed inversely.

The only thing needed to define a homomorphism is that the set is closed under the binary operation; this is why we have defined homomorphism for the most general structures, i.e., groupoids.

Isomorphic groups

Definition 2. *If there exists an isomorphism between two groups, these groups are isomorphic.*

Example 3. *The two groups* $\mathbb{Z}_5^\times$ *and* $\mathbb{Z}_4^+$ *are isomorphic. We have even found two distinct isomorphisms:* φ_1 *and* φ_2*.*

Isomorphic groups have the same order.

Fundamental properties of homomorphisms (1/2)

Theorem 4. *Let* φ *be a homomorphism from a group* $G = (M, \circ_G)$ *to a group* $H = (N, \circ_H)$.

The group $\varphi(G) = (\varphi(M), \circ_H)$ *is a subgroup of* H.

Proof. Each element in $\varphi(G)$ can be written as $\varphi(x)$ for some $x \in M$.

- For all $x, y, z \in M$ we have that

$$(\varphi(x) \circ_H \varphi(y)) \circ_H \varphi(z) = \varphi(x \circ_G y) \circ_H \varphi(z) = \varphi((x \circ_G y) \circ_G z) =$$
$$= \varphi(x \circ_G (y \circ_G z)) = \varphi(x) \circ_H \varphi(y \circ_G z) = \varphi(x) \circ_H (\varphi(y) \circ_H \varphi(z))$$

- Denote by e_G the neutral element in G. Then $\varphi(e_G)$ is the neutral element in $\varphi(G)$ because, for all $x \in M$, we have $\varphi(e_G) \circ_H \varphi(x) = \varphi(e_G \circ_G x) = \varphi(x)$.

- It can be shown similarly that the inverse of $\varphi(x)$ is $\varphi(x^{-1})$. □

Fundamental properties of homomorphisms (2/2)

Consequences of the previous theorem and its proof:

- A homomorphism always maps the neutral element of one group to the neutral element of the other group.

- Inverse elements are preserved as well: $\varphi(x^{-1}) = \varphi(x)^{-1}$.

Example 5.

$$\begin{array}{rcl} \varphi : \mathbb{Z}_4^+ & \to & \mathbb{Z}_8^+ \\ n & \mapsto & 2n \end{array}$$

is a homomorphism and $\varphi(\mathbb{Z}_4^+)$ *is the subgroup* $\{0, 2, 4, 6\} \leq \mathbb{Z}_8^+$.

. . . up to isomorphism (1/4)

Isomorphic groups are in fact identical, they differ only in the names of their elements (as we have seen in the case of groups $\mathbb{Z}_4^+$ and $\mathbb{Z}_5^\times$).

If we say that there exists one group with a certain property up to isomorphism, it means that all groups with this property are isomorphic to each other.

We prove three well-known statements of this kind.

Theorem 6.
- *Any two infinite cyclic groups are isomorphic.*
- *For each $n \in \mathbb{N}$, any two cyclic groups of order n are isomorphic.*

Proof: hint. Let $G = \langle a \rangle$ be a cyclic group with generator a.

We show that an arbitrary infinite cyclic group is isomorphic to the group $(\mathbb{Z}, +)$, and that an arbitrary cyclic group of order n is isomorphic to $\mathbb{Z}_n^+$.

The rest follows from the transitivity of the relation "to be isomorphic". □

$(\mathbb{Z}, +)$ and $\mathbb{Z}_n^+$ are the only cyclic groups up to isomorphism.

. . . up to isomorphism (2/4)

The Klein group is the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \circ)$, where

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$$

and $\circ$ is the component-wise addition modulo 2: e.g., $(1,0) \circ (1,1) = (0,1)$.

The Klein group is not cyclic and thus cannot be isomorphic to $\mathbb{Z}_4^+$!

It is possible to show this (try it, it is easy):

Theorem 7. *There exists only two groups of order 4 which are not isomorphic.*

$\mathbb{Z}_4^+$ and the Klein group are the only two groups of order 4 up to isomorphism.

. . . up to isomorphism (3/4)

The symmetric group $\mathcal{S}_n$ of the set of all permutations over $\{1, 2, 3, \ldots, n\}$ with the operation of composition.

- A (n-)permutation is a bijection of the set $\{1, 2, 3, \ldots, n\}$ to itself, so $\mathcal{S}_n$ is the set of bijections on $\{1, 2, 3, \ldots, n\}$.

- Each permutation $\pi \in \mathcal{S}_n$ can be defined by listing its values:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}.$$

  The first row could by deleted, and so, e.g., $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ is the permutation swapping elements 3 and 4.

- Composition of permutations: $(1\ 2\ 4\ 3\ 5) \circ (2\ 1\ 3\ 5\ 4) = (2\ 1\ 4\ 5\ 3)$.

- The composition of permutations is associative, the permutation $(1\,2\,3\ \cdots n)$ is the neutral element, and the inverse element is the inverse permutation. Hence, $\mathcal{S}_n$ is a group of order $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

. . . up to isomorphism(4/4)

Subgroups of the symmetric group $\mathcal{S}_n$ are called groups of permutations.

Example 8. *The permutation* $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ *swapping the elements* 3 *and* 4 *generates a subgroup of* $\mathcal{S}_5$ *containing two elements:* $(1\ 2\ 4\ 3\ 5)$ *and* $(1\ 2\ 3\ 4\ 5)$.

The structure of the subgroups of $\mathcal{S}_n$ is very (in some sense maximally) rich:

Theorem 9 (Cayley)**.** *Each finite group is isomorphic to some group of permutations.*

Proof: hint only for interested. Let a be an element of a group G of order n with a binary operation $\circ$.

Put $\pi_a(x) = a \circ x$. Since in any group we can divide uniquely, π_a is a bijection and thus a permutation. The desired monomorphism is the mapping defined for each element a in this way: $\varphi(a) = \pi_a$. □

Application of group theory in cryptography

Diffie-Hellman Key Exchange

Discrete logarithm problem

The standard logarithm (in base α) of the number β is the solution of the equation

$$\alpha^x = \beta \quad \text{in the group } (\mathbb{R}, \cdot).$$

Definition 10 (Discrete logarithm problem in $\mathbb{Z}_p^\times$)**.** *Let us consider the group $\mathbb{Z}_p^\times$, α one of its generator and β one of its element.*

To solve the discrete logarithm problem means to find the integer $1 \leq x \leq p-1$ such that

$$\alpha^x \equiv \beta \ (mod\ p)$$

The discrete logarithm?

No reasonably fast algorithm solving the discrete logarithm problem is known.

But rising to the power in $\mathbb{Z}_p^\times$ can be done effectively.

The speed of the best known algorithms is roughly proportional to $\sqrt{p}$, i.e., for p having its binary representation 1024 bits long, such algorithm makes approximately 2^{512} operations.

Thus we obtain a one-way function that can be used for **asymmetric cipher**:

- Find $\beta \equiv \alpha^x \pmod{p}$ is easy, knowing x, α and p;
- Find x, knowing β, α and p is very difficult

In **RSA** (**R**ivest-**S**hamir-**A**dleman) cryptosystem, the one way function "multiplying of primes" is used:

- Multiplication of primes is easy and fast, while prime factorization of the result is very difficult.

Alice	**Bob**
chooses private key $a \in \{2, \dots, p-2\}$	chooses private key $b \in \{2, \dots, p-2\}$
computes public key $A \equiv \alpha^a \bmod p$	computes public key $B \equiv \alpha^b \bmod p$

exchange of public keys A and B

$\longleftrightarrow$

computes $k_{AB} \equiv B^a \bmod p$	computes $k_{AB} \equiv A^b \bmod p$

RSA

Alice

Initialization: she finds two large prime numbers p and q,
she computes $n = p \cdot q$ and $\psi(n) = (p-1)(q-1)$,
she chooses $e \in \{1, 2, \dots, \psi(n) - 1\}$ so that $\gcd(e, \psi(n)) = 1$,
she computes the private key d so that $d \cdot e = 1 \mod \psi(n)$.
She sends the public key $k_{pub} = (n, e)$ to Bob.

Bob

Bob wants to send the message x.
He encrypts the message $y = x^e \mod n$ and sends y to Alice.

Alice

Alice decrypts the message by $x = y^d \mod n$.

Diffie-Hellman Key Exchange

Initialization: Alice finds some large prime number p and some generator α of the group $\mathbb{Z}_p^\times$.

She publishes p and α. (Finding a large prime and a generator are not easy tasks!)

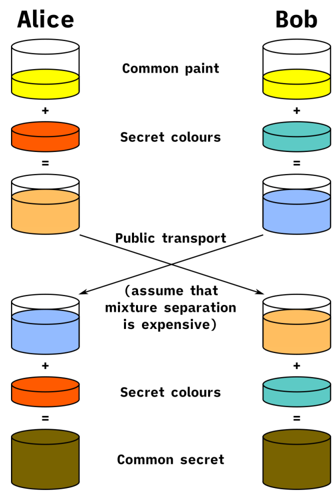
Principle

Diffie-Hellman Key Exchange is built on the following facts:

- Rising to the power in $\mathbb{Z}_p^\times$ is commutative, and so the value of k_{AB} is the same for both Alice and Bob:

$$k_{AB} \equiv (\alpha^b)^a \equiv \alpha^{ab} \mod p$$
$$k_{AB} \equiv (\alpha^a)^b \equiv \alpha^{ab} \mod p.$$

- Rising to the power is not computationally complex (square & multiply algorithm).
- The inverse operation to rising to the power (the discrete logarithm) is computationally exhausting.



Discrete logarithm in general

The discrete logarithm problem can be defined in an arbitrary cyclic group.

Definition 11 (problem of discrete logarithm in group $G = (M, \cdot)$)**.** *Let $G = (M, \cdot)$ be a cyclic group of order n, α one of its generators and β one of its an element.*

To solve the discrete logarithm problem means to find the integer $1 \leq x \leq n$ s.t.

$$\alpha^x = \beta.$$

If we use additive notation:

Definition 12 (problem of discrete logarithm in group $G = (M, +)$)**.** *Let $G = (M, +)$ be a cyclic group of order n, α one of its generators and β one of its element.*

To solve the discrete logarithm problem means to find the integer $1 \leq k \leq n$ s.t.

$$k \times \alpha = \beta.$$

The discrete logarithm is not always complicated

Consider the group $\mathbb{Z}_p^+$.

It is a cyclic group of prime order p, and each positive $\alpha < p-1$ is its generator. The problem of discrete logarithm in this group has the form of the equation

$$k\alpha \equiv \beta \pmod{p}.$$

We can solve it easily: we find the inverse of α in the group $\mathbb{Z}_p^\times$ (by polynomial EEA, see the following lectures), and the solution is $k = \beta\alpha^{-1} \pmod{p}$.

Example 13. *Let $p = 11$, $\alpha = 3$ and $\beta = 5$. We want to find k such that $k \cdot 3 \equiv 5\ (mod\ 11)$.*

We easily verify that in $\mathbb{Z}_{11}^\times$ we have $3^{-1} = 4$, and thus $k = 5 \cdot 4\ (mod\ 11) = 9$.

Question 14. *We know that groups $\mathbb{Z}_p^\times$ and $\mathbb{Z}_{p-1}^+$ are isomorphic.*

Is this a problem for the Diffie-Hellman algorithm?