Mathematics for Informatics

Multivariate optimization
(lecture 2 of 12)

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What shall we do today?

- Multivariate optimization:
    - Gradient
    - Tangent plane
    - Critical points on two or more variables
    - Hessian (matrix)

Gradient of a function

The gradient of a function $f(x_1, x_2, \ldots, x_n)$ at the (n-dimensional) point $b \in \mathbb{R}^n$ is the n-dimensional vector function $\nabla f(b)$ defined by

$$\nabla f(b) = \left(\frac{\partial f}{\partial x_1}(b), \frac{\partial f}{\partial x_2}(b), \ldots, \frac{\partial f}{\partial x_n}(b) \right).$$

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Find the gradient of the function $f(x, y) = x^2 + xy + y^2$ *at the point* $(1, 1)$.

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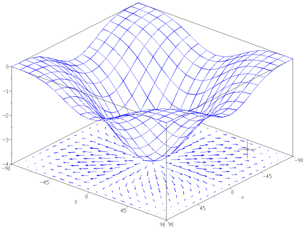
Geometrical meaning: the gradient points is the direction of the greatest rate of increase of the function. Its magnitude equals the rate of increase.



Gradient and the directional derivative

We saw that the partial derivative with respect to x at the point a is equal to the slope of tangent line at this point in direction parallel to the x-axis.

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Example

If we are on the graph of the fonction $f(x, y) = x^2 + xy + y^2$ *at the point* $(1, 1)$ *and we start moving in the direction parallel to the* x*-axis, i.e., in the direction of the vector* $(1, 0)$*, we will go "uphill" under the angle* $\arctan 3$ *since*

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What will be the slope if we move in the direction of a general vector $\vec{v}$?

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Theorem

Given a function $f(x) : \mathbb{R}^n \to \mathbb{R}$*, a point* $a \in \mathbb{R}^n$ *and a* ***unit*** *vector* $\vec{v} \in \mathbb{R}^n$*, the derivative in the direction of the vector* $\vec{v}$ *is the dot product of the gradient and* $\vec{v}$*, i.e,* $\nabla f(a_1, a_2, \ldots, a_n) \cdot \vec{v}$.

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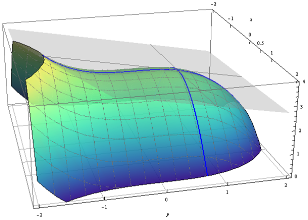
$$z = \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) \; + \; \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0) \; + \; f(x_0, y_0).$$



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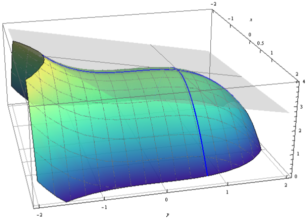

Example

Find the tangent plane to $f(x, y) = x^2 + xy + y^2$ *at* $(1, 1)$.

Critical points – two variables

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The first class of these points can be found as a solution of

$$\nabla f(x, y) = (0, 0)$$

which leads to the system of two equations for two variables

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) & = & 0 \\ \frac{\partial f}{\partial y}(x, y) & = & 0 \end{cases}.$$

Critical points – more variables

For an n-variable function $f(x_1, x_2, \dots, x_n)$ the situation is analogous:
The critical points of $f(x_1, x_2, \dots, x_n)$ are points satisfying the equation

$$\nabla f(x_1, x_2, \dots, x_n) = 0$$

i.e., points satisfying the system of n equations for n variables

$$\begin{cases} \dfrac{\partial f}{\partial x_1}(x_1, x_2, \dots, x_n) & = & 0 \\ \dfrac{\partial f}{\partial x_2}(x_1, x_2, \dots, x_n) & = & 0 \\ & \vdots & \\ \dfrac{\partial f}{\partial x_n}(x_1, x_2, \dots, x_n) & = & 0 \end{cases},$$

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or where the gradient does not exist.
(Instead of a tangent plane, we have a tangent hyperplane.)

Critical points – example

Example

Find all critical points of

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which always exists.

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which always exists. Thus the critical points are the solution of the system

$$\begin{cases} x_3 + 2x_1 & = & 0 \\ -1 + x_3 + 2x_2 & = & 0 \\ x_1 + x_2 + 6x_3 & = & 0 \end{cases},$$

which, using the standard procedure for a system of linear equations, gives us the only solution $\left(\frac{1}{20}, \frac{11}{20}, \frac{-1}{10}\right)$.

Type of a critical point (1 of 4)

In the one dimensional case, we can use the second derivative to decide the type of the critical point.

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Do we have something similar for more variables? What is the second derivative?

Type of a critical point (2 of 4)

The analogue of the second derivative is the following.

Definition

For a function $f(x_1, x_2, \ldots, x_n)$ we define the Hessian matrix as

$$\nabla^2 f(x_1, x_2, \ldots, x_n) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(x_1, \ldots, x_n) & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(x_1, \ldots, x_n) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1}(x_1, \ldots, x_n) & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}(x_1, \ldots, x_n) \end{pmatrix}$$

assuming that all the derivatives exist.

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We would like to construct rules like "If $f''(x_0) > 0$, then the critical point x_0 is the point of strict minimum".

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5. *indefinite otherwise.*

Type of a critical point (4 of 4)

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ has all second partial derivative continuous at a critical point $b \in \mathbb{R}^n$, then

1. *if $\nabla^2 f(b)$ is positively definite, then b is a point of strict local minimum;*
2. *if $\nabla^2 f(b)$ is negatively definite, then b is a point of strict local maximum;*
3. *if $\nabla^2 f(b)$ is indefinite, then b is a saddle point.*

Sylvester's criterion on definiteness

For an $n \times n$ dimensional **symmetric** matrix A we define the principal minors:

- M_1 is the upper left 1-by-1 corner of A,
- M_2 is the upper left 2-by-2 corner of A,
- ...
- M_n is the upper left n-by-n corner of A.

Theorem

Let $A \in \mathbb{R}^{n,n}$ be a symmetric matrix.

- *A is positively definite if and only if the determinants of all principal minors are positive.*
- *A is negatively definite if and only if the determinant of M_i is negative for odd i and positive for even i.*

Example

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Find all minima and maxima of the function

$$f(x,y) = \frac{3x^4 - 4x^3 - 12x^2 + 18}{12(1+4y^2)}.$$

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Solution: The critical points are $(-1,0)$, $(0,0)$ and $(2,0)$; they are a saddle point, a point of maximum and a point of minimum, respectively.