Mathematics for Informatics

Homomorphisms, Application in cryptography (lecture 6 of 12)

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Outline

- Homomorphisms
- Application of groups theory in cryptography

The same groups and distinct elements (1/5)

$\mathbb{Z}_5^\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\mathbb{Z}_4^+$	0	1	2	3
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neutral element: 0

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Aren't these two groups in fact the same group differing only in the "names" of their elements?

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- Now, it remains to rename only 2 and 3; we can check that both remaining possibilities work; we choose, for instance, $3 \leftrightarrow 1$ and $2 \leftrightarrow 3$.
- It suffices to reorder the rows. . . and we have the Cayley table of $\mathbb{Z}_4^+$.

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This renaming is actually an **injective** mapping of the set $\{1, 2, 3, 4\}$ **onto** the set $\{0, 1, 2, 3\}$; let us denote it φ_1:

$$\varphi_1(1) = 0, \qquad \varphi_1(2) = 3, \qquad \varphi_1(3) = 1, \qquad \varphi_1(4) = 2.$$

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Would all bijections do the same job? And if not, what makes these two so special?

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Let us rename the elements of the group $\mathbb{Z}_5^\times$ according to the bijection φ_3:

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The bijection φ_3 does not give rise to the same structure of the group $\mathbb{Z}_4^+$; only φ_1 and φ_2 have this property.

The same groups and distinct elements (5/5)

The desired property, which only the bijections φ_1 and φ_2 have, is the following:

$$\text{for all } n, m \in \{1, 2, 3, 4\}, \text{ we have } \varphi(n \times_5 m) = \varphi(n) +_4 \varphi(m),$$

where $\times_5$ denotes the operation in the group $\mathbb{Z}_5^\times$, and $+_4$ the one in the group $\mathbb{Z}_4^+$.

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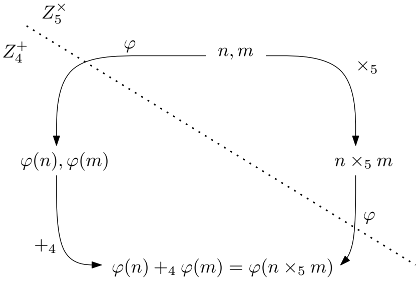
*In words: If we apply the operation $\times_5$ to two arbitrary elements of the group $\mathbb{Z}_5^\times$ and then we send the result to $\mathbb{Z}_4^+$ by φ, we obtain the same result as when we first transform by φ the elements to $\mathbb{Z}_4^+$ and **then** apply the operation $+_4$.*



Homomorphism and isomorphism

Definition

Let $G = (M, \circ_G)$ and $H = (N, \circ_H)$ be two groupoids. The mapping $\varphi : M \to N$ is a homomorphism from G to H if

$$\text{for all } x, y \in M, \text{ we have } \quad \varphi(x \circ_G y) = \varphi(x) \circ_H \varphi(y).$$

If, moreover, φ is injective (resp. surjective, resp. bijective) we say that φ is a monomorphism (resp. epimorphism, resp. isomorphism).

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A homomorphism preserves the structure given by the binary operation: the result is the same if we first apply the operation and then the homomorphism or if we proceed inversely.

The only thing needed to define a homomorphism is that the set is closed under the binary operation; this is why we have defined homomorphism for the most general structures, i.e., groupoids.

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Isomorphic groups have the same order.

Fundamental properties of homomorphisms (1/2)

Theorem

Let φ be a homomorphism from a group $G = (M, \circ_G)$ to a group $H = (N, \circ_H)$. The group $\varphi(G) = (\varphi(M), \circ_H)$ is a subgroup of H.

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- For all $x, y, z \in M$ we have that

$$(\varphi(x) \circ_H \varphi(y)) \circ_H \varphi(z) = \varphi(x \circ_G y) \circ_H \varphi(z) = \varphi((x \circ_G y) \circ_G z) =$$
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- Denote by e_G the neutral element in G. Then $\varphi(e_G)$ is the neutral element in $\varphi(G)$ because, for all $x \in M$, we have $\varphi(e_G) \circ_H \varphi(x) = \varphi(e_G \circ_G x) = \varphi(x)$.

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- It can be shown similarly that the inverse of $\varphi(x)$ is $\varphi(x^{-1})$. $\square$

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Consequences of the previous theorem and its proof:

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Example

$$\begin{aligned} \varphi : \mathbb{Z}_4^+ &\to \mathbb{Z}_8^+ \\ n &\mapsto 2n \end{aligned}$$

is a homomorphism and $\varphi(\mathbb{Z}_4^+)$ *is the subgroup* $\{0, 2, 4, 6\} \leq \mathbb{Z}_8^+$.

. . . up to isomorphism (1/4)

Isomorphic groups are in fact identical, they differ only in the names of their elements (as we have seen in the case of groups $\mathbb{Z}_4^+$ and $\mathbb{Z}_5^\times$).
If we say that there exists one group with a certain property up to isomorphism, it means that all groups with this property are isomorphic to each other.
We prove three well-known statements of this kind.

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- *Any two infinite cyclic groups are isomorphic.*
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Proof: hint.

Let $G = \langle a \rangle$ be a cyclic group with generator a.
We show that an arbitrary infinite cyclic group is isomorphic to the group $(\mathbb{Z}, +)$, and that an arbitrary cyclic group of order n is isomorphic to $\mathbb{Z}_n^+$.
The rest follows from the transitivity of the relation “to be isomorphic”. $\square$

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$(\mathbb{Z}, +)$ and $\mathbb{Z}_n^+$ are the only cyclic groups up to isomorphism.

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The Klein group is the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \circ)$, where

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$$

and $\circ$ is the component-wise addition modulo 2: e.g., $(1,0) \circ (1,1) = (0,1)$.

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The Klein group is not cyclic and thus cannot be isomorphic to $\mathbb{Z}_4^+$!
It is possible to show this (try it, it is easy):

Theorem

There exists only two groups of order 4 which are not isomorphic.

$\mathbb{Z}_4^+$ and the Klein group are the only two groups of order 4 up to isomorphism.

. . . up to isomorphism (3/4)

The symmetric group $\mathcal{S}_n$ of the set of all permutations over $\{1, 2, 3, \ldots, n\}$ with the operation of composition.

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- Each permutation $\pi \in \mathcal{S}_n$ can be defined by listing its values:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}.$$

The first row could by deleted, and so, e.g., $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ is the permutation swapping elements 3 and 4.

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- Composition of permutations: $(1\ 2\ 4\ 3\ 5) \circ (2\ 1\ 3\ 5\ 4) = (2\ 1\ 4\ 5\ 3)$.

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- A (n-)permutation is a bijection of the set $\{1, 2, 3, \dots, n\}$ to itself, so $\mathcal{S}_n$ is the set of bijections on $\{1, 2, 3, \dots, n\}$.
- Each permutation $\pi \in \mathcal{S}_n$ can be defined by listing its values:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}.$$

  The first row could by deleted, and so, e.g., $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ is the permutation swapping elements 3 and 4.
- Composition of permutations: $(1\ 2\ 4\ 3\ 5) \circ (2\ 1\ 3\ 5\ 4) = (2\ 1\ 4\ 5\ 3)$.
- The composition of permutations is associative, the permutation $(1\ 2\ 3\ \cdots n)$ is the neutral element, and the inverse element is the inverse permutation. Hence, $\mathcal{S}_n$ is a group of order $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

... up to isomorphism(4/4)

Subgroups of the symmetric group $\mathcal{S}_n$ are called groups of permutations.

Example

The permutation $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ *swapping the elements* 3 *and* 4 *generates a subgroup of* $\mathcal{S}_5$ *containing two elements:* $(1\ 2\ 4\ 3\ 5)$ *and* $(1\ 2\ 3\ 4\ 5)$.

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The structure of the subgroups of $\mathcal{S}_n$ is very (in some sense maximally) rich:

Theorem (Cayley)

Each finite group is isomorphic to some group of permutations.

Proof: hint only for interested.

Let a be an element of a group G of order n with a binary operation $\circ$.
Put $\pi_a(x) = a \circ x$. Since in any group we can divide uniquely, π_a is a bijection and thus a permutation. The desired monomorphism is the mapping defined for each element a in this way: $\varphi(a) = \pi_a$. □

Discrete logarithm problem

The standard logarithm (in base α) of the number β is the solution of the equation

$$\alpha^x = \beta \quad \text{in the group } (\mathbb{R}, \cdot).$$

Definition (Discrete logarithm problem in $\mathbb{Z}_p^\times$)

Let us consider the group $\mathbb{Z}_p^\times$, α one of its generator and β one of its element. To solve the discrete logarithm problem means to find the integer $1 \leq x \leq p-1$ such that

$$\alpha^x \equiv \beta \ (mod\ p)$$

The discrete logarithm?

No reasonably fast algorithm solving the discrete logarithm problem is known. But rising to the power in $\mathbb{Z}_p^\times$ can be done effectively.

The speed of the best known algorithms is roughly proportional to $\sqrt{p}$, i.e., for p having its binary representation 1024 bits long, such algorithm makes approximately 2^{512} operations.

Thus we obtain a one-way function that can be used for **asymmetric cipher**:

- Find $\beta \equiv \alpha^x \pmod{p}$ is easy, knowing x, α and p;
- Find x, knowing β, α and p is very difficult

In **RSA** (**R**ivest-**S**hamir-**A**dleman) cryptosystem, the one way function "multiplying of primes" is used:

- Multiplication of primes is easy and fast, while prime factorization of the result is very difficult.

RSA

Alice

Initialization: she finds two large prime numbers p and q,
she computes $n = p \cdot q$ and $\psi(n) = (p-1)(q-1)$,
she chooses $e \in \{1, 2, \ldots, \psi(n) - 1\}$ so that $\gcd(e, \psi(n)) = 1$,
she computes the private key d so that $d \cdot e = 1 \mod \psi(n)$.

She sends the public key $k_{pub} = (n, e)$ to Bob.

Bob

Bob wants to send the message x.
He encrypts the message $y = x^e \mod n$ and sends y to Alice.

Alice

Alice decrypts the message by $x = y^d \mod n$.

Diffie-Hellman Key Exchange

Initialization: Alice finds some large prime number p and some generator α of the group $\mathbb{Z}_p^\times$.
She publishes p and α. (Finding a large prime and a generator are not easy tasks!)

Alice	**Bob**
chooses private key $a \in \{2, \dots, p-2\}$	chooses private key $b \in \{2, \dots, p-2\}$
computes public key $A \equiv \alpha^a \bmod p$	computes public key $B \equiv \alpha^b \bmod p$

$$\xleftrightarrow{\text{exchange of public keys } A \text{ and } B}$$

computes $k_{AB} \equiv B^a \bmod p$	computes $k_{AB} \equiv A^b \bmod p$

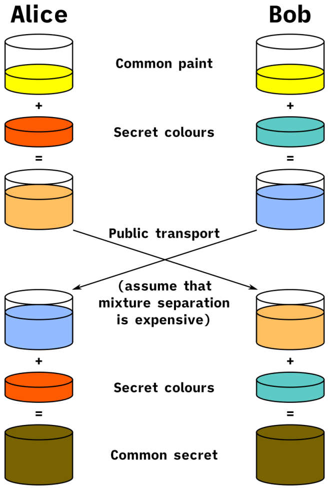
Principle

Diffie-Hellman Key Exchange is built on the following facts:

- Rising to the power in $\mathbb{Z}_p^\times$ is commutative, and so the value of k_{AB} is the same for both Alice and Bob:

$$\begin{aligned} k_{AB} &\equiv (\alpha^b)^a \equiv \alpha^{ab} \mod p \\ k_{AB} &\equiv (\alpha^a)^b \equiv \alpha^{ab} \mod p. \end{aligned}$$

- Rising to the power is not computationally complex (square & multiply algorithm).
- The inverse operation to rising to the power (the discrete logarithm) is computationally exhausting.



Discrete logarithm in general

The discrete logarithm problem can be defined in an arbitrary cyclic group.

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If we use additive notation:

Definition (problem of discrete logarithm in group $G = (M, +)$)

Let $G = (M, +)$ be a cyclic group of order n, α one of its generators and β one of its element.
To solve the discrete logarithm problem means to find the integer $1 \leq k \leq n$ s.t.

$$k \times \alpha = \beta.$$

The discrete logarithm is not always complicated

Consider the group $\mathbb{Z}_p^+$.
It is a cyclic group of prime order p, and each positive $\alpha < p-1$ is its generator.
The problem of discrete logarithm in this group has the form of the equation

$$k\alpha \equiv \beta \pmod{p}.$$

We can solve it easily: we find the inverse of α in the group $\mathbb{Z}_p^\times$ (by polynomial EEA, see the following lectures), and the solution is $k = \beta\alpha^{-1} \pmod{p}$.

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Example

Let $p = 11$, $\alpha = 3$ and $\beta = 5$. We want to find k such that $k \cdot 3 \equiv 5 \ (mod\ 11)$.

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Question

We know that groups $\mathbb{Z}_p^\times$ and $\mathbb{Z}_{p-1}^+$ are isomorphic.
Is this a problem for the Diffie-Hellman algorithm?