

Mathematics for Informatics

Rings and fields
(lecture 7 of 12)

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Fall 2021/2022

created: November 1, 2021, 11:21

Outline

- A more complicated group: Points in elliptic curves;
- Rings and fields;
- Finite fields in general;
- Binary fields.

Calculation on elliptic curves

On elliptic curves, we use points (x, y) with x and y being the residue classes modulo some prime number p .

The operation, which is usually denoted by $+$, is defined as follows:

Definition (Don't even try to memorize it!)

For two points $\mathcal{P} = (x_1, y_1)$ and $\mathcal{Q} = (x_2, y_2)$, we define $\mathcal{P} + \mathcal{Q} = (x_3, y_3)$ as:

$$x_3 \equiv s^2 - x_1 - x_2 \pmod{p}$$

$$y_3 \equiv s(x_1 - x_3) - y_1 \pmod{p}$$

where

$$s = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1} \pmod{p} & \text{if } \mathcal{P} \neq \mathcal{Q} \\ \frac{3x_1^2 + a}{2y_1} \pmod{p} & \text{if } \mathcal{P} = \mathcal{Q}. \end{cases}$$

The parameter a is taken from the equation of a given elliptic curve $y^2 \equiv x^3 + ax + b \pmod{p}$, with $a, b \in \mathbb{Z}_p$, which must be fulfilled by all points. The neutral element \mathcal{O} is “artificially” defined such that it has the properties of the neutral element.

Elliptic curves over the field of real numbers (1/3)

To better understand what is happening on elliptic curves, we consider points (x, y) from the *continuous* plane \mathbb{R}^2 .

Definition

An *elliptic curve* is the set of points given by the equation

$$y^2 = x^3 + ax + b,$$

where the real coefficients a, b satisfy $4a^3 + 27b^2 \neq 0$.

The group operation can be defined by geometrical means (see [Wolfram Demonstrations Project](#)).

Elliptic curves over the field of real numbers (2/3)

When we add two points $\mathcal{P} = (x_1, y_1)$ and $\mathcal{Q} = (x_2, y_2)$, we plot a line through \mathcal{P} and \mathcal{Q} and we look for intersections of this line and the elliptic curve.

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Or if you don't like the drawings and really want the exact formula:

- If $\mathcal{P} \neq \mathcal{Q}$, the line going through \mathcal{P} and \mathcal{Q} has equation

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2) + y_2.$$

- If $\mathcal{P} = \mathcal{Q} = (x_1, y_1)$, a tangent line to the elliptic curve at $\mathcal{P} = \mathcal{Q}$ has slope given by the derivative of the curve $y = \sqrt{x^3 + ax + b}$ at x_1 , which is

$$\frac{3x^2 + a}{2\sqrt{x^3 + ax + b}} = \{\text{for } x = x_1\} = \frac{3x_1^2 + a}{2y_1}, \text{ and thus the line has equation}$$

$$y = \frac{3x_1^2 + a}{2y_1}(x - x_1) + y_1.$$

Elliptic curves over the field of real numbers (3/3)

Now, we are looking for intersections of some line $y = sx + d$, where s is the slope and $d \in \mathbb{R}$, and the elliptic curve $y^2 = x^3 + ax + b$.

This leads to solving the equation

$$(sx + d)^2 = x^3 + ax + b,$$

i.e., a polynomial equation of order 3, which can have 1, 2 or 3 real roots (we have always at least one solution; namely the points \mathcal{P} and \mathcal{Q}).

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In general, the solution of the equation is obtained in this way: For two points $\mathcal{P} = (x_1, y_1)$ and $\mathcal{Q} = (x_2, y_2)$ define $\mathcal{P} + \mathcal{Q} = (x_3, y_3)$ as:

$$x_3 = s^2 - x_1 - x_2 \quad \text{and} \quad y_3 = s(x_1 - x_3) - y_1$$

where

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Elliptic curves over a general field

The previous three slides show how to define a group over the set of points in the plane $\mathbb{R} \times \mathbb{R}$.

Generally, instead of the set \mathbb{R} we could take an arbitrary **field**, e.g., the field \mathbb{Z}_p , where the discrete logarithm problem is difficult to solve.

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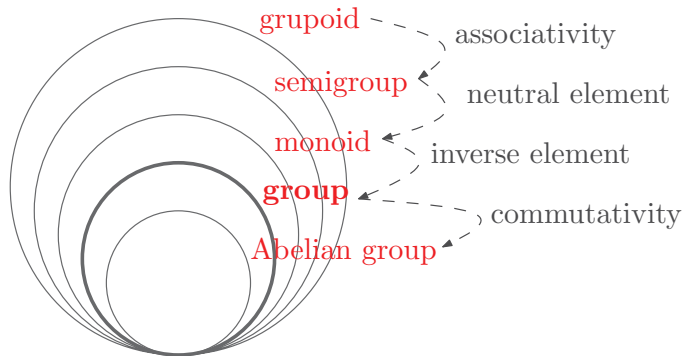
Generally, instead of the set \mathbb{R} we could take an arbitrary **field**, e.g., the field \mathbb{Z}_p , where the discrete logarithm problem is difficult to solve.

Field, what is that?

Generally speaking, it is a set with two binary operations (usually called *addition* and *multiplication*) which allow us to define analogues of common arithmetical operations such as subtracting, dividing, rising to the power, logarithm, etc.

Sets with one binary operation

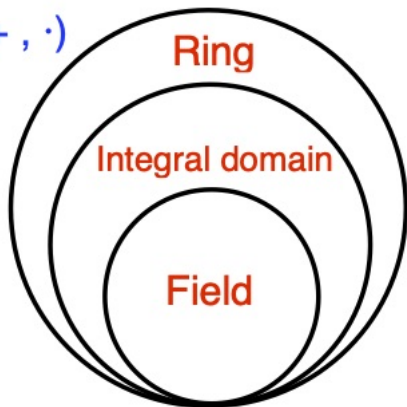
A nonempty set M with a binary operation \cdot (resp. $+$ for the additive notation).



Sets with two binary operations

For more sophisticated arithmetical operations with numbers we need **both** addition and multiplication.

$(M, +, \cdot)$



Definition of a ring

Definition (Ring)

Let M be a nonempty set, and $+$ and \cdot two binary operations. We say that $R = (M, +, \cdot)$ is a **ring** if the following holds:

- $(M, +)$ is an **Abelian group**,
- (M, \cdot) is a **monoid**,
- both left and right **distributive law** hold:

$$(\forall a, b, c \in M) \text{ we have: } a(b + c) = ab + ac \quad \wedge \quad (b + c)a = ba + ca.$$

We respect the standard convention that the multiplication has a higher priority than the addition (and use the concatenation, instead of \cdot , when it is clear from the context).

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Sometimes, (M, \cdot) is required to be only a semigroup (or even a groupoid).

Terminology

Let $R = (M, +, \cdot)$ be a ring.

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- Inside the ring **we can define the operation subtraction** by

$$a - b := a + (-b).$$

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- The set $(\mathbb{R}^{n,n}, +, \cdot)$ of square real matrices with the usual addition and multiplication is a ring; the zero element is the zero matrix.
- The set of all polynomials (with complex / real / integer coefficients) is a ring; the zero element is the zero polynomial $p(x) = 0$.

Basic properties of rings

In an arbitrary ring $(M, +, \cdot)$, the following holds.

- Left and right distributive law for subtracting, i.e.,

$$c(b - a) = cb - ca.$$

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Indeed:

$$a \cdot 0 = a(a - a) = aa - aa = 0.$$

□

Integral domain

Definition (zero divisors)

Let $R = (M, +, \cdot)$ be a ring. Two arbitrary **nonzero** elements $a, b \in M$ such that

$$a \cdot b = 0$$

are called **zero divisors**.

Definition (integral domain)

A commutative ring **without** zero divisors is called an **integral domain**.

Examples of integral domains

- $(\mathbb{Z}, +, \cdot)$ is an integral domain.

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- Each number ring $(M, +, \cdot)$, where $M \subset \mathbb{C}$ and $+$ and \cdot are classical, is an integral domain.
- The ring $(\mathbb{R}^{n,n}, +, \cdot)$ is **not** an integral domain for $n \geq 2$, because it is not commutative; moreover, it has zero divisors:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition of field

Definition (field)

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Why do we have to remove the zero element?

Because the zero has no inverse (with respect to the multiplication), i.e., it is not possible to divide by zero: $0^{-1} = ?!$.

We can **divide** by all other elements of the field!

dividing := multiplying by the inverse element

$$\frac{a}{b} := a \cdot b^{-1} \quad \text{for } b \neq 0.$$

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- The ring of rational numbers $(\mathbb{Q}, +, \cdot)$ is a field. Moreover, it is the smallest number field (with the common arithmetical operations).
- The smallest field is the so-called **trivial field** $(\{0, 1\}, +, \cdot)$, with operations given by the following tables:

$+$	0	1
0	0	1
1	1	0

and

\cdot	0	1
0	0	0
1	0	1

The first table corresponds to the bit operation XOR and the latter to AND, or, alternatively, to the addition and multiplication modulo 2.

Some properties

In each field all usual arithmetical operations are defined:

addition, subtraction, multiplication, division, and all operations derived from them such as *rising to the power, root extractions, logarithm, ...*

Using the trivial field we have all these operations over one bit. Later we will show how to extend them to any number of bits.

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Proof.

Since the multiplicative group of the field $(M \setminus \{0\}, \cdot)$ is closed under multiplication, for all nonzero a, b it holds that their product $a \cdot b \in M \setminus \{0\}$ is again nonzero. □

Homomorphism and isomorphism

Definition

A mapping f from the ring (resp. field) R_1 to the ring (resp. field) R_2 is a *homomorphism* if f is a homomorphism of the corresponding additive and multiplicative groupoids (resp. groups).

If, moreover, f is bijective (injective and surjective), it is an *isomorphism*.

Finite fields

A field with finite number of elements is called **finite**.

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An example of finite field is the set (of residue classes modulo p)

$$\mathbb{Z}_p = \{0, 1, \dots, p-1\}$$

with operations modulo a **prime** p (see previous lectures).

E.g., for $p = 5$ we obtain the field with following operations:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

and

\cdot_5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Additive group $(\mathbb{Z}_p, +_p)$

- The order of the group $\mathbb{Z}_p^+ = (\mathbb{Z}_p, +_p)$ is the prime number p .
- Each nonzero element is a generator (this holds for all groups with prime order).
- $(\mathbb{Z}_p, +_p)$ is a group even when p is not prime.

Multiplicative group $(\mathbb{Z}_p \setminus \{0\}, \cdot_p)$

- The order of the group $\mathbb{Z}_p^\times = (\mathbb{Z}_p \setminus \{0\}, \cdot_p)$ is $p - 1$ and this is never a prime number for $p \neq 3$!
- \mathbb{Z}_p^\times is cyclic (i.e., there exists a generator).
- The number of generators depends on $p - 1$, and is equal to the number of numbers coprime to $p - 1$, i.e., $\varphi(p - 1)$.
- If k with $k < p$ divides $p - 1$, then there exists a subgroup in \mathbb{Z}_p^\times of order k and it contains just the elements for which $a^k = 1$.

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*Any finite field has **order** p^n , where p is a prime number and n is a positive natural number. The prime number p is called the **characteristic** of the field. Furthermore, all fields of order p^n are isomorphic. Additionally, the multiplicative group of a finite field is cyclic.*

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Consequence: There are no fields of order 6, 10, 12, 14, ...

If we chose $p = 2$ and $n = 8$, we obtain the field providing us with arithmetic on 1 byte (8 bits)!

Symmetric cryptography

Secure exchange of longer text performed by asymmetric ciphers (RSA, Diffie-Hellman and others) is not effective.

That is why the **symmetric ciphers** are used: symmetric ciphers assume that (and take advantage of) Alice and Bob share some secret key. Asymmetric ciphers are used only for exchanging this private key.

A very common method is the block cipher called **A**dvanced **E**ncryption **S**tandard (AES).

Here we get acquainted with the mathematics underlying this method.

AES block cipher

The text we want to securely transfer is divided into (e.g.) blocks having 8 bits. Then, these blocks are encoded using the shared key so that the decoding can be easily made using the same key.

This cipher AES is based on the fact that arithmetic operations with $n = 8$ bits can be understood as operations in a finite field with 2^n elements for $n = 8$. The fields with 2^n elements are called **binary fields** and are denoted $GF(2^n)$ (as *Galois Fields*).

We now explain how to define addition and multiplication in these fields.

The wrong way

Consider a field $GF(2^8)$. Each element can be represented as an 8 bit string, e.g., 11010110 , 01100011 , etc.

Addition: Addition can be defined component-wise modulo 2. i.e.

$$11010110 + 01100011 = (1 +_2 0)(1 +_2 1) \cdots (0 +_2 1) = 10110101.$$

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Multiplication must be defined in a different way!

Rings of polynomials over a ring / field

In order to be able to add, subtract, and multiply a polynomial of the form $\sum_i a_i x^i$, we only need to know how to add, subtract, and multiply the coefficients.

In general, we can construct a ring of polynomials over an arbitrary ring or field similar to the one we know from real or complex numbers.

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Definition

Let K be a ring. The *commutative ring of polynomials over K* , denoted $K[x]$, is the set of polynomials with coefficients in K together with operations of addition and multiplication defined as:

$$\begin{aligned} \left(\sum_{i=0}^n a_i x^i \right) + \left(\sum_{i=0}^n b_i x^i \right) &= \sum_{i=0}^n (a_i + b_i) x^i; \\ \left(\sum_{i=0}^n a_i x^i \right) \cdot \left(\sum_{i=0}^m b_i x^i \right) &= \sum_{i=0}^{n+m} \left(\sum_{j+k=i} a_j b_k \right) x^i. \end{aligned}$$

Irreducible polynomial

Definition

Let K be a field and $P(x) \in K[x]$ be of degree at least 1.

We say that $P(x)$ is *irreducible over K* if, for any two polynomials $A(x)$ and $B(x)$ from $K[x]$, it holds that

$$A(x) \cdot B(x) = P(x) \quad \Rightarrow \quad (\text{degree of } A(x) = 0 \quad \vee \quad \text{degree of } B(x) = 0).$$

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Irreducible polynomials are primes among polynomials!

Their definition is analogous as well as their properties.

Example: Whereas $x^2 + 1$ is irreducible over the field \mathbb{Q} , the polynomial $x^2 - 1 = (x + 1)(x - 1)$ is not.

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Remark: $x^2 + 1$ is irreducible over the field \mathbb{Q} , but not over the field \mathbb{Z}_2 , where the coefficients are added and multiplied modulo 2:

$$x^2 + 1 = (x + 1)(x + 1) = x^2 + 2x + 1.$$

Irreducible polynomial as a modulus

We define **modulo polynomial** as:

$A(x) \pmod{P(x)} :=$ the remainder of the division of $A(x)$ by $P(x)$.

The result is always a polynomial of degree less than the degree of $P(x)$.

Irreducible polynomial as a modulus

We define **modulo polynomial** as:

$$A(x) \pmod{P(x)} := \text{the remainder of the division of } A(x) \text{ by } P(x).$$

The result is always a polynomial of degree less than the degree of $P(x)$.

Example: for $A(x) = x^3$ and $P(x) = x^2 + 1$ we have $A(x) = x(x^2 + 1) + (-x)$ and thus

$$x^3 \equiv -x \pmod{x^2 + 1}.$$

If $P(x)$ is irreducible (with respect to the field from which the coefficients are taken), the remainders after division by $P(x)$ form a group (if we again remove the zero polynomial).

Field $GF(2^4)$

The elements $GF(2^4)$ are represented as polynomials of order at most 3 with coefficients h_i from the field \mathbb{Z}_2 :

$$h_3x^3 + h_2x^2 + h_1x + h_0 \approx (h_3h_2h_1h_0)_2.$$

Thus, the 16 elements of $GF(2^4)$ are:

$$0, 1, x, x + 1, \dots, x^3 + x^2 + x + 1$$

or

$$0000, 0001, 0010, 0011, \dots, 1111.$$

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Addition component-wise modulo 2:

$$(x^3 + x + 1) + (x^2 + x + 1) = x^3 + x^2.$$

Field $GF(2^4)$ – multiplication

Multiplication modulo a **chosen** irreducible polynomial, e.g., $x^4 + x + 1$.

Example: multiplication $A(x) \cdot B(x)$ for $A(x) = x^3 + x^2 + 1 \approx 1101$ and $B(x) = x^2 + x \approx 0110$.

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- 1 Multiply $A(x) \cdot B(x)$ classically and rewrite coefficients **mod 2**:

$$A(x) \cdot B(x) = x^5 + 2x^4 + x^3 + x^2 + x = x^5 + x^3 + x^2 + x.$$

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- 2 Find the remainder after division by $P(x)$. Since

$$x^5 = x(x^4 + x + 1) + (x^2 + x), \quad \text{it holds } x^5 \equiv x^2 + x \pmod{x^4 + x + 1},$$

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$$x^5 + x^3 + x^2 + x \equiv (x^2 + x) + (x^3 + x^2 + x) \equiv x^3 \pmod{x^4 + x + 1}.$$

Hence we get that $1101 \cdot 0110 = 1000 \pmod{x^4 + x + 1}$.

AES in field $GF(2^8)$

According to the specification of AES, the multiplication is done modulo

$$x^8 + x^4 + x^3 + x + 1.$$

Construction of a finite field

In general, we construct a finite field $GF(p^k)$ using polynomials as follows.

Let $m(x) \in \mathbb{Z}_p[x]$ be an irreducible polynomial of degree k .

$$GF(p^k) = \left(\{q(x) \in \mathbb{Z}_p[x] : \deg(q) < k\}, +, \cdot \text{ mod } m(x) \right).$$