NIE-MPI: Tutorial 4

`created: November 3, 2021, 13:02`

4.1 Algebra

Exercise 4.1. Find out which of the following groupoids is a group and which is an Abelian group.

(a) $(\mathbb{N}, +)$, where $\mathbb{N}$ is the set of natural (non-negative) integers;

(b) $(\mathbb{Z}, +)$, where $\mathbb{Z}$ is the set of integers;

(c) $(\mathbb{R}, \cdot)$, where $\mathbb{R}$ is the set of real numbers;

(d) $(\mathbb{C}, +)$, where $\mathbb{C}$ is the set of complex numbers.

Exercise 4.2. Which of the following couples forms a group?

(a) $(\mathbb{Z}^-, +)$, where $\mathbb{Z}^-$ is the set of negative integers;

(b) $(\mathbb{Q}, +)$, where $\mathbb{Q}$ is the set of rational numbers;

(c) $(\mathbb{Q}_0^+, +)$, where $\mathbb{Q}_0^+$ is the set of non-negative rational numbers;

(d) $(\{-1, 1\}, +)$;

(e) $(\{-1, 1\}, \cdot)$.

Exercise 4.3. Which of the following sets with the operation $\circ$ defined for all a, b by

$$a \circ b = a^b$$

forms a groupoid/semigroup/group/Abelian group?

(a) $(\mathbb{Q}, \circ)$;

(b) $(\mathbb{N}, \circ)$;

(c) $(\{-1, 1\}, \circ)$.

Exercise 4.4. Which of the following sets of complex square matrices creates a group with the common matrix multiplication?

(a) Real matrices;

(b) Regular real matrices;

(c) Regular diagonal real matrices;

(d) Upper (lower) regular triangular real matrices.

Exercise 4.5. Let us consider the set $M = \mathbb{Z} \times \mathbb{Z} \times \{1, -1\}$ and the operation $\otimes$ on it defined by:

$$(k_1, \ell_1, 1) \otimes (k_2, \ell_2, \varepsilon) = (k_1 + k_2, \ell_1 + \ell_2, \varepsilon)$$
$$(k_1, \ell_1, -1) \otimes (k_2, \ell_2, \varepsilon) = (k_1 + k_2, \ell_1 + \ell_2, -\varepsilon)$$

for $\varepsilon \in \{1, -1\}$. Is $(M, \otimes)$ a group?

Exercise 4.6. Let us define the operations $\sqcap$ and $\triangle$ on $\mathbb{R}$ as:

$$a \sqcap b = a + b + 1 \quad \text{and} \quad a \triangle b = a + b + ab$$

for arbitrary $a, b \in \mathbb{R}$.

Prove that

(a) $(\mathbb{R}, \sqcap)$ is an Abelian group;

(b) $(\mathbb{R}, \triangle)$ is not a group.

Exercise 4.7. Let ρ be a plane. For arbitrary points $A, B \in \rho$ let $A \star B$ be the central point of the segment AB. Is $(\rho, \star)$ a group?

Exercise 4.8. Let for arbitrary $a, b \in \mathbb{N}$ be $a \circ b = a^b$ (see Exercise 4.3). Find all triples $a, b, c \in \mathbb{N}$ for which we have $(a \circ b) \circ c = a \circ (b \circ c)$.

Exercise 4.9. Let us consider the set $\mathbb{N}$ and the operation $\wedge$ such that for arbitrary $a, b \in \mathbb{N}$ we have $a \wedge b$ equal to the greatest common divisor of a, b. Is $(N, \wedge)$ a group?

Exercise 4.10. Decide whether the following table is the Cayley table of a group:

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

How can we recognize that it is Cayley table of an Abelian group?