MPI - Lecture 6

Outline

- Homomorphisms
- Application of groups theory in cryptography

Homomorphisms

Motivation

The same groups and distinct elements (1/5)

$\mathbb{Z}_5^\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

order: 4 [2mm] subgroups: $\{1\}$, $\{1,4\}$, $\{1,2,3,4\}$ [2mm] neutral element: 1 [2mm] inverse elements: $1^{-1}=1,\quad 2^{-1}=3,$ $3^{-1}=2,\quad 4^{-1}=4.$

$\mathbb{Z}_4^+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

order: 4 [2mm] subgroups: $\{0\}$, $\{0,2\}$, $\{0,1,2,3\}$ [2mm] neutral element: 0 [2mm] inverse elements: $-0=0,\quad -1=3,$ $-2=2,\quad -3=1.$ Aren't these two groups in fact the same group differing only in the "names" of their elements?

$\mathbb{Z}_5^\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\mathbb{Z}_4^+$	0	3	1	2
0	0	3	1	2
3	3	2	0	1
1	1	0	2	3
2	2	1	3	0

$\mathbb{Z}_4^+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Let us try to rename the elements of the group $\mathbb{Z}_5^\times$ so to get $\mathbb{Z}_4^+$:

- The neutral element has very special and unique properties: we rename 1 to 0.
- If the complete structure should be preserved, then the only two-elements subgroup $\{1, 4\}$ (in $\mathbb{Z}_5^\times$) must correspond to the subgroup $\{0, 2\}$ (in $\mathbb{Z}_4^+$): we map $4 \leftrightarrow 2$.
- Now, it remains to rename only 2 and 3; we can check that both remaining possibilities work; we choose, for instance, $3 \leftrightarrow 1$ and $2 \leftrightarrow 3$.
- It suffices to reorder the rows. . . and we have the Cayley table of $\mathbb{Z}_4^+$.

We have found a way to rename the elements in one table to gain an exact copy of the other table (after rearranging rows and columns).

This renaming is actually an **injective** mapping of the set $\{1, 2, 3, 4\}$ **onto** the set $\{0, 1, 2, 3\}$; let us denote it φ_1:

$$\varphi_1(1) = 0, \qquad \varphi_1(2) = 3, \qquad \varphi_1(3) = 1, \qquad \varphi_1(4) = 2.$$

We have pointed out that the mapping φ_2 works as well:

$$\varphi_2(1) = 0, \qquad \varphi_2(2) = 1, \qquad \varphi_2(3) = 3, \qquad \varphi_2(4) = 2.$$

Would all bijections do the same job? And if not, what makes these two so special?

The same groups and distinct elements (4/5)

Let us rename the elements of the group $\mathbb{Z}_5^\times$ according to the bijection φ_3:

$$\varphi_3(1) = 0, \qquad \varphi_3(2) = 3, \qquad \varphi_3(3) = 2, \qquad \varphi_3(4) = 1.$$

$\mathbb{Z}_5^\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\varphi_3(\mathbb{Z}_5^\times)$	0	3	2	1
0	0	3	2	1
3	3	1	0	2
2	2	0	1	3
1	1	2	3	0

$\mathbb{Z}_4^+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

The resulting table is not the Cayley table of the group $\mathbb{Z}_4^+$, because, e.g., $3 + 3 \pmod 4 \neq 1$.

The bijection φ_3 does not give rise to the same structure of the group $\mathbb{Z}_4^+$; only φ_1 and φ_2 have this property.

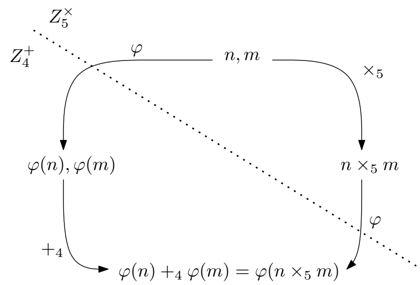
The same groups and distinct elements (5/5)

The desired property, which only the bijections φ_1 and φ_2 have, is the following:

$$\text{for all } n, m \in \{1, 2, 3, 4\}, \text{ we have } \varphi(n \times_5 m) = \varphi(n) +_4 \varphi(m),$$

where $\times_5$ denotes the operation in the group $\mathbb{Z}_5^\times$, and $+_4$ the one in the group $\mathbb{Z}_4^+$.

In words: If we apply the operation $\times_5$ to two arbitrary elements of the group $\mathbb{Z}_5^{\times}$ and then we send the result to $\mathbb{Z}_4^{+}$ by φ, we obtain the same result as when we first transform by φ the elements to $\mathbb{Z}_4^{+}$ and **then** apply the operation $+_4$.



Definition and properties

Homomorphism and isomorphism

Definition 1. *Let $G = (M, \circ_G)$ and $H = (N, \circ_H)$ be two groupoids. The mapping $\varphi : M \to N$ is a homomorphism from G to H if*

$$\text{for all } x, y \in M, \text{ we have } \quad \varphi(x \circ_G y) = \varphi(x) \circ_H \varphi(y).$$

If, moreover, φ is injective (resp. surjective, resp. bijective) we say that φ is a monomorphism (resp. epimorphism, resp. isomorphism).

A homomorphism preserves the structure given by the binary operation: the result is the same if we first apply the operation and then the homomorphism or if we proceed inversely.

The only thing needed to define a homomorphism is that the set is closed under the binary operation; this is why we have defined homomorphism for the most general structures, i.e., groupoids.

Isomorphic groups

Definition 2. *If there exists an isomorphism between two groups, these groups are isomorphic.*

Example 3. *The two groups $\mathbb{Z}_5^{\times}$ and $\mathbb{Z}_4^{+}$ are isomorphic. We have even found two distinct isomorphisms: φ_1 and φ_2.*

Isomorphic groups have the same order.

Fundamental properties of homomorphisms (1/2)

Theorem 4. *Let φ be a homomorphism from a group $G = (M, \circ_G)$ to a group $H = (N, \circ_H)$.*

The group $\varphi(G) = (\varphi(M), \circ_H)$ is a subgroup of H.

Proof. Each element in $\varphi(G)$ can be written as $\varphi(x)$ for some $x \in M$.

- For all $x, y, z \in M$ we have that

$$(\varphi(x) \circ_H \varphi(y)) \circ_H \varphi(z) = \varphi(x \circ_G y) \circ_H \varphi(z) = \varphi((x \circ_G y) \circ_G z) =$$
$$= \varphi(x \circ_G (y \circ_G z)) = \varphi(x) \circ_H \varphi(y \circ_G z) = \varphi(x) \circ_H (\varphi(y) \circ_H \varphi(z))$$

- Denote by e_G the neutral element in G. Then $\varphi(e_G)$ is the neutral element in $\varphi(G)$ because, for all $x \in M$, we have $\varphi(e_G) \circ_H \varphi(x) = \varphi(e_G \circ_G x) = \varphi(x)$.

- It can be shown similarly that the inverse of $\varphi(x)$ is $\varphi(x^{-1})$. $\square$

Fundamental properties of homomorphisms (2/2)

Consequences of the previous theorem and its proof:

- A homomorphism always maps the neutral element of one group to the neutral element of the other group.

- Inverse elements are preserved as well: $\varphi(x^{-1}) = \varphi(x)^{-1}$.

Example 5.

$$\begin{array}{rcl} \varphi : \mathbb{Z}_4^+ & \to & \mathbb{Z}_8^+ \\ n & \mapsto & 2n \end{array}$$

is a homomorphism and $\varphi(\mathbb{Z}_4^+)$ is the subgroup $\{0, 2, 4, 6\} \leq \mathbb{Z}_8^+$.

. . . up to isomorphism (1/4)

Isomorphic groups are in fact identical, they differ only in the names of their elements (as we have seen in the case of groups $\mathbb{Z}_4^+$ and $\mathbb{Z}_5^\times$).

If we say that there exists one group with a certain property up to isomorphism, it means that all groups with this property are isomorphic to each other.

We prove three well-known statements of this kind.

Theorem 6.
- *Any two infinite cyclic groups are isomorphic.*
- *For each $n \in \mathbb{N}$, any two cyclic groups of order n are isomorphic.*

Proof: hint. Let $G = \langle a \rangle$ be a cyclic group with generator a.

We show that an arbitrary infinite cyclic group is isomorphic to the group $(\mathbb{Z}, +)$, and that an arbitrary cyclic group of order n is isomorphic to $\mathbb{Z}_n^+$.

The rest follows from the transitivity of the relation "to be isomorphic". □

$(\mathbb{Z}, +)$ and $\mathbb{Z}_n^+$ are the only cyclic groups up to isomorphism.

. . . up to isomorphism (2/4)

The Klein group is the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \circ)$, where

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$$

and $\circ$ is the component-wise addition modulo 2: e.g., $(1,0) \circ (1,1) = (0,1)$.

The Klein group is not cyclic and thus cannot be isomorphic to $\mathbb{Z}_4^+$!

It is possible to show this (try it, it is easy):

Theorem 7. *There exists only two groups of order 4 which are not isomorphic.*

$\mathbb{Z}_4^+$ and the Klein group are the only two groups of order 4 up to isomorphism.

. . . up to isomorphism (3/4)

The symmetric group $\mathcal{S}_n$ of the set of all permutations over $\{1, 2, 3, \ldots, n\}$ with the operation of composition.

- A (n-)permutation is a bijection of the set $\{1, 2, 3, \ldots, n\}$ to itself, so $\mathcal{S}_n$ is the set of bijections on $\{1, 2, 3, \ldots, n\}$.

- Each permutation $\pi \in \mathcal{S}_n$ can be defined by listing its values:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}.$$

  The first row could by deleted, and so, e.g., $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ is the permutation swapping elements 3 and 4.

- Composition of permutations: $(1\ 2\ 4\ 3\ 5) \circ (2\ 1\ 3\ 5\ 4) = (2\ 1\ 4\ 5\ 3)$.

- The composition of permutations is associative, the permutation $(1\,2\,3\ \cdots n)$ is the neutral element, and the inverse element is the inverse permutation. Hence, $\mathcal{S}_n$ is a group of order $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

. . . up to isomorphism(4/4)

Subgroups of the symmetric group $\mathcal{S}_n$ are called groups of permutations.

Example 8. *The permutation* $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ *swapping the elements* 3 *and* 4 *generates a subgroup of* $\mathcal{S}_5$ *containing two elements:* $(1\ 2\ 4\ 3\ 5)$ *and* $(1\ 2\ 3\ 4\ 5)$*.*

The structure of the subgroups of $\mathcal{S}_n$ is very (in some sense maximally) rich:

Theorem 9 (Cayley)**.** *Each finite group is isomorphic to some group of permutations.*

Proof: hint only for interested. Let a be an element of a group G of order n with a binary operation $\circ$.

Put $\pi_a(x) = a \circ x$. Since in any group we can divide uniquely, π_a is a bijection and thus a permutation. The desired monomorphism is the mapping defined for each element a in this way: $\varphi(a) = \pi_a$. $\square$