

Mathematics for Informatics

Multivariate optimization
(lecture 2 of 12)

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What shall we do today?

- Multivariate optimization:
 - Gradient
 - Tangent plane
 - Critical points on two or more variables
 - Hessian (matrix)

Gradient of a function

The **gradient** of a function $f(x_1, x_2, \dots, x_n)$ at the (n -dimensional) point $b \in \mathbb{R}^n$ is the n -dimensional vector function $\nabla f(b)$ defined by

$$\nabla f(b) = \left(\frac{\partial f}{\partial x_1}(b), \frac{\partial f}{\partial x_2}(b), \dots, \frac{\partial f}{\partial x_n}(b) \right).$$

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Example

Find the gradient of the function $f(x, y) = x^2 + xy + y^2$ at the point $(1, 1)$.

Gradient of a function

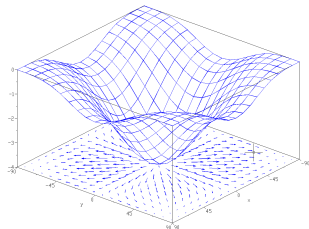
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Example

Find the gradient of the function $f(x, y) = x^2 + xy + y^2$ at the point $(1, 1)$.

Geometrical meaning: the gradient points in the direction of the greatest rate of increase of the function. Its magnitude equals the rate of increase.



Gradient and the directional derivative

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Theorem

Given a function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, a point $a \in \mathbb{R}^n$ and a **unit** vector $\vec{v} \in \mathbb{R}^n$, the *derivative in the direction of the vector \vec{v}* is the dot product of the gradient and \vec{v} , i.e., $\nabla f(a_1, a_2, \dots, a_n) \cdot \vec{v}$.

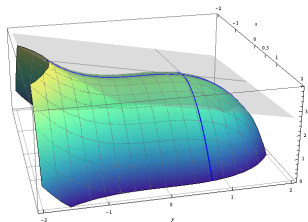
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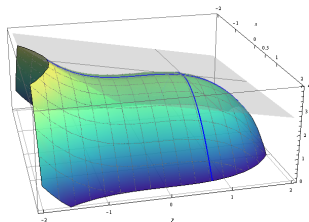
$$z = \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0) + f(x_0, y_0).$$



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Example

Find the tangent plane to $f(x, y) = x^2 + xy + y^2$ at $(1, 1)$.

Critical points – two variables

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The first class of these points can be found as a solution of

$$\nabla f(x, y) = (0, 0)$$

which leads to the system of two equations for two variables

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = 0 \\ \frac{\partial f}{\partial y}(x, y) = 0 \end{cases} .$$

Critical points – more variables

For an n -variable function $f(x_1, x_2, \dots, x_n)$ the situation is analogous:
 The **critical points** of $f(x_1, x_2, \dots, x_n)$ are points satisfying the equation

$$\nabla f(x_1, x_2, \dots, x_n) = 0$$

i.e., points satisfying the system of n equations for n variables

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x_1}(x_1, x_2, \dots, x_n) = 0 \\ \frac{\partial f}{\partial x_2}(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_1, x_2, \dots, x_n) = 0 \end{array} \right. ,$$

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or where the gradient does not exist.

(Instead of a tangent plane, we have a **tangent hyperplane**.)

Critical points – example

Example

Find all critical points of

$$f(x_1, x_2, x_3) = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2,$$

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$$\nabla f(x_1, x_2, x_3) = (x_3 + 2x_1, -1 + x_3 + 2x_2, x_1 + x_2 + 6x_3)$$

which always exists.

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which always exists. Thus the critical points are the solution of the system

$$\begin{cases} x_3 + 2x_1 & = & 0 \\ -1 + x_3 + 2x_2 & = & 0 \\ x_1 + x_2 + 6x_3 & = & 0 \end{cases},$$

which, using the standard procedure for a system of linear equations, gives us the only solution $\left(\frac{1}{20}, \frac{11}{20}, \frac{-1}{10}\right)$.

Type of a critical point (1 of 4)

In the one dimensional case, we can use the second derivative to decide the type of the critical point.

Theorem

Let x_0 be a critical point of a function $f(x)$ such that $f'(x_0) = 0$ and $f''(x_0)$ exists.

- If $f''(x_0) > 0$, then the function is **convex** at x_0 , and x_0 is a point of a **minimum**.
- If $f''(x_0) < 0$, then the function is **concave** at x_0 , and x_0 is a point of a **maximum**.
- If $f''(x_0) = 0$, then x_0 may be a minimum, maximum, inflection point, ...

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Do we have something similar for more variables? What is the second derivative?

Type of a critical point (2 of 4)

The analogue of the second derivative is the following.

Definition

For a function $f(x_1, x_2, \dots, x_n)$ we define the *Hessian matrix* as

$$\nabla^2 f(x_1, x_2, \dots, x_n) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x_1, \dots, x_n) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x_1, \dots, x_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x_1, \dots, x_n) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x_1, \dots, x_n) \end{pmatrix}$$

assuming that all the derivatives exist.

Type of a critical point (3 of 4)

We would like to construct rules like “If $f''(x_0) > 0$, then the critical point x_0 is the point of minimum”.

But to say that the matrix is “positive” is problematic . . . Let us use a different notion.

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- ① *positively definite* if for all non-zero vectors $a \in \mathbb{R}^n$ it holds that $aAa^T > 0$;

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- (v) *indefinite* otherwise.

Type of a critical point (4 of 4)

Theorem

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ has all second partial derivative continuous at a critical point $b \in \mathbb{R}^n$, then

- (i) if $\nabla^2 f(b)$ is positively definite, then b is a point of local minimum;
- (ii) if $\nabla^2 f(b)$ is negatively definite, then b is a point of local maximum;
- (iii) if $\nabla^2 f(b)$ is indefinite, then b is a saddle point.

Sylvester's criterion on definiteness

For an $n \times n$ dimensional **symmetric** matrix A we define the **principal minors**:

- M_1 is the upper left 1-by-1 corner of A ,
- M_2 is the upper left 2-by-2 corner of A ,
- ...
- M_n is the upper left n -by- n corner of A .

Theorem

Let $A \in \mathbb{R}^{n,n}$ be a symmetric matrix.

- A is positively definite if and only if the determinants of all principal minors are positive.
- A is negatively definite if and only if the determinant of M_i is negative for odd i and positive for even i .

Example

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Find all minima and maxima of the function

$$f(x, y) = \frac{3x^4 - 4x^3 - 12x^2 + 18}{12(1 + 4y^2)}.$$

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Solution: The critical points are $(-1, 0)$, $(0, 0)$ and $(2, 0)$; they are a saddle point, a point of maximum and a point of minimum, respectively.

