BIE-DML - Discrete Mathematics and Logic

Tutorial 1

Formulas, truth valuation, logical laws

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Winter semester 2023/2024

updated: 31/10/2023, 10:29

1.1 Introduction

Definition 1.1 (Formulas of PL)**. Formulas of propositional logic** are:

1. All elementary formulas $A, B, C, \ldots$.
2. If A, B are propositional formulas, then so are $(\neg A), (A \wedge B), (A \vee B), (A \Rightarrow B), (A \Leftrightarrow B)$.
3. Only formulas obtained by **final** usage of rules 1 and 2 are propositional formulas.

Remark 1.2. We usually omit unnecessary brackets, i.e., we simply write

$$A \wedge B, \quad \neg A \Rightarrow B, \quad (A \Rightarrow B) \vee \neg((A \wedge B) \vee B)$$

instead of

$$(A \wedge B),\ ((\neg A) \Rightarrow B)\,,\ ((A \Rightarrow B) \vee (\neg((A \wedge B) \vee B)))$$

Definition 1.3 (Logical connectives)**.**

Name	**Notation**	**Formal Usage**	**How to read it**
negation	$\neg$	$\neg A$	not A
conjunction	$\wedge$	$A \wedge B$	A and B
disjunction	$\vee$	$A \vee B$	A or B
implication	$\Rightarrow$	$A \Rightarrow B$	A implies B, if A (then) B
equivalence	$\Leftrightarrow$	$A \Leftrightarrow B$	A if and only if (iff) B, A when and only when B

1.1.1 Truth valuation

Truth valuation of a propositional formula can be determined from the truth values of all elementary formulas using the rules for logical connectives (summed up below):

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

Definition 1.4. We say that:

- Formula F is **true** in truth valuation v if $v(F) = 1$.
- Formula F is **false** in truth valuation v if $v(F) = 0$.
- Formula F is **satisfiable** if there is truth valuation v where $v(F) = 1$.
- Formula F is **unsatisfiable** if there is no truth valuation v where $v(F) = 1$.
- Formula F is a **tautology** (denoted by $\top$) if $v(F) = 1$ for every valuation v (i.e., it is always true).
- Formula F is a **contradiction** (denoted by $\bot$) if $v(F) = 0$ for every valuation v (i.e., it is always false).

Definition 1.5 ($A \Leftrightarrow\!\!\!\!\!\equiv B$).
Formulas A, B are **logically equivalent** ($A \models\!\!\!\dashv B$) if and only if $v(A) = v(B)$ for every valuation v.

Remark 1.6. $A \models\!\!\!\dashv B$ iff $A \Leftrightarrow B$ is a tautology.

Theorem 1.7 (Properties of tautology and contradiction)**.**

1. $\neg\top \models\!\!\!\dashv \bot$ ... *(The negation of a tautology is a contradiction.)*
2. $\neg\bot \models\!\!\!\dashv \top$ ... *(The negation of a contradiction is a tautology.)*
3. *A tautology is satisfiable.*
4. *A contradiction is unsatisfiable.*

Moreover, for any formula of PL A holds:

1. $A \wedge \top \models\!\!\!\dashv A$ ... *identity law for* $\wedge$
2. $A \vee \top \models\!\!\!\dashv \top$ ... *identity law for* $\vee$
3. $A \wedge \bot \models\!\!\!\dashv \bot$ ... *universal bound law for* $\wedge$
4. $A \vee \bot \models\!\!\!\dashv A$ ... *universal bound law for* $\vee$

1.1.2 Logical Laws

Theorem 1.8. *Let A, B, C be formulas of PL. Then*

1. $A \vee \neg A \models\!\!\!\dashv \top$ ... *law of excluded middle*
2. $\neg(A \wedge \neg A) \models\!\!\!\dashv \top$ ... *law of non-contradiction*
3. $\neg\neg A \Leftrightarrow A \models\!\!\!\dashv \top$ ... *law of double negation*
4. $A \wedge A \models\!\!\!\dashv A$ ... *idempotent law for* $\wedge$
5. $A \vee A \models\!\!\!\dashv A$ ... *idempotent law for* $\vee$
6. $A \wedge B \models\!\!\!\dashv B \wedge A$ ... *commutative law for* $\wedge$
7. $A \vee B \models\!\!\!\dashv B \vee A$ ... *commutative law for* $\vee$
8. $(A \wedge B) \wedge C \models\!\!\!\dashv A \wedge (B \wedge C)$ ... *associative law for* $\wedge$
9. $(A \vee B) \vee C \models\!\!\!\dashv A \vee (B \vee C)$ ... *associative law for* $\vee$
10. $(A \wedge B) \vee C \models\!\!\!\dashv (A \vee C) \wedge (B \vee C)$ ... *distributive law*
11. $(A \vee B) \wedge C \models\!\!\!\dashv (A \wedge C) \vee (B \wedge C)$ ... *distributive law*
12. $A \wedge (A \vee B) \models\!\!\!\dashv A$ ... *absorption law for* $\vee$
13. $A \vee (A \wedge B) \models\!\!\!\dashv A$ ... *absorption law for* $\wedge$
14. $\neg(A \wedge B) \models\!\!\!\dashv \neg A \vee \neg B$ ... *de Morgan law*
15. $\neg(A \vee B) \models\!\!\!\dashv \neg A \wedge \neg B$ ... *de Morgan law*

16. $A \Rightarrow B \boxminus \neg B \Rightarrow \neg A$ ... *law of contraposition*

17. $A \Rightarrow B \boxminus \neg A \vee B$ ... *"golden rule"*

18. $\neg(A \Rightarrow B) \boxminus A \wedge \neg B$ ... *"silver rule"*

19. $A \Leftrightarrow B \boxminus (A \Rightarrow B) \wedge (B \Rightarrow A)$ ... *meaning of the equivalence*

1.1.3 Necessary and Sufficient Conditions

Assume we have an implication $A \Rightarrow B$. If A is false then the implication is automatically true – this case is of no interest to us. If the implication should be true and so is A, then B must be also true. So we can say that it **suffices** for A to be true for B to be also true, i.e., A is a sufficient condition for B. On the other hand, if A is true then, **necessarily**, B has to be true – B is a necessary condition for A.

In an implication $A \Rightarrow B$, A is a **sufficient condition** for B and B is a **necessary condition** for A.

If we have an equivalence $A \Leftrightarrow B$, then B is a sufficient and necessary condition for A, as well as A being a necessary and sufficient condition for B (there is a symmetry in an equivalence).

Example 1.9. Without oxygen, there would be no human life; hence oxygen is a **necessary** condition for the existence of human beings. We can rephrase it as "If there is human life then there must be oxygen" and formalize $H \Rightarrow O$ with H denoting "There is human life" and O denoting "There is oxygen".
(Alternatively, we can say ""If there was no oxygen then there would be no human life", $\neg O \Rightarrow \neg H$. This is an equivalent formulation which demonstrates the law of contraposition.)

Example 1.10. Some scientists think that if there is oxygen then there must be (some kind of) life; so oxygen might also be a **sufficient** condition for (some kind of) life to exist. We would say "If there is oxygen then there must be (some kind of) life", and write $O \Rightarrow L$, with O meaning "There is oxygen" and L meaning "There is (some kind of) life".

1.2 Exercises

Exercise 1.1. Use these elementary propositions and the proposed letters

a) Today it's Wednesday. (W)

b) Today it's Monday. (M)

c) We have a class today. (C)

d) Jane has a cat. (A)

e) Jane has a dog. (O)

f) An animal barks. (B)

g) An animal is a dog. (D)

to formalize these sentences:

i) Today it's not Wednesday.

ii) We don't have a class today.

iii) Today it's Wednesday and we have a class.

iv) Jane has a cat and a dog.

v) Jane has a cat but she doesn't have a dog.

vi) Jane has a cat or a dog. - not exclusive or!!!

vii) Today it's Monday or we have a class.

viii) If an animal barks then it's a dog.

ix) An animal barks only if it's a dog.

x) An animal is a dog only if it barks.

xi) * We have a class today unless it's Monday. (= If it's **not** Monday today then we have a class)

xii) Today it's Wednesday if and only if we have a class.

Remark 1.11. There are several ways of expressing implication. Some are quite straightforward: *if A then B, B if A, B whenever A* all mean *If A is true then B is true.*Others are more confusing: *A only if B* means *if A then B* (for example "Nick will eat his lunch only if he is very hungry." means "If Nick will eat his lunch then he is very hungry."), and *A unless B* actually means *if not B then A* (for example "You cannot drive unless you are 18." means "If you are not 18 then you cannot drive.").

Exercise 1.2. The sentences below contain negation(s). Try to simplify them and formulate them as positive declarations, where possible.
Example: Negative: It's not true that 2 is less than 1. Positive: 2 is greater than or equal to 1.

a) A number x is not even.

b) A number x is not greater than 3.

c) I do not have two children.

d) It's not the case that Mars is the closest planet to the Sun.

e) It's not true that Bob is not a good student.

f) * It's not true that I haven't decided not to go to the party.

Exercise 1.3. Formalize the sentences below using these two elementary propositions and the given letters:

- T: "I take the tram."
- S: "I take the subway."

a) I take the tram and the subway.

b) I take the tram or the subway.

c) If I take the tram then I take the subway too.

d) If I take the tram then I don't take the subway.

e) I take the tram if and only if I don't take the subway.

f) I take neither the tram nor the subway.

g) I take the tram or the subway but not both.

h) * Unless I take the subway, I don't take the tram.

Exercise 1.4. Identify the elementary propositions and formalize the following sentences using the letters proposed.
Example: "Cats don't bark." (B)
Denote the statement "Cats bark" by B. Then "Cats don't bark" can be represented by $\neg B$.

a) Jane likes cats (C) and dogs (D).

b) Tonight I will read a book (B) or watch TV (T).

c) If Mr. Jones is happy (R), Mrs. Jones is happy (S), and if Mr. Jones is unhappy, Mrs. Jones is unhappy.

d) The bribe will be paid (B) if and only if the goods are delivered (G).

e) John goes to the movies (M) only if a comedy is playing (C).

f) It's not the case that students don't like logic. (L)

g) Max (M) and Charles (C) will go to the party but neither Tamara (T) nor Lucy (L) will come.

h) Charles (C) will not come without Lucy. (L)

i) You can either have soup (P) or salad (S) but not both. (exclusive disjunction)

j) * If you don't like to fly (F) then you can visit some countries (V) only if you go by ship (S).

k) ** You can't live in the Czech Republic (L) unless you are a Czech citizen (C) or you have a visa (V).

Exercise 1.5. List all subformulas and construct truth tables for the following formulas. Decide whether they are satisfiable, contradictions or tautologies.

a) $(\neg A \Rightarrow B) \vee A$

b) $(A \wedge B) \Rightarrow \neg(A \Leftrightarrow \neg B)$

c) $(A \vee \neg C) \Rightarrow B$

d) $((A \Rightarrow B) \wedge A) \Rightarrow B$

e) $(\neg(A \wedge B) \Rightarrow C) \Leftrightarrow \neg C$

f) $(A \Rightarrow (B \wedge C)) \Rightarrow ((A \Rightarrow B) \wedge (A \Rightarrow C))$

g) * $((A \Rightarrow B) \vee \neg(C \wedge D)) \iff (\neg(D \Rightarrow B) \wedge (A \vee C))$

Exercise 1.6. Prove the validity of logical laws in Theorem 1.7.

a) The negation of a tautology is a contradiction. $(\neg\top \boxminus \bot)$

b) The negation of a contradiction is a tautology. $(\neg\bot \boxminus \top)$

c) A tautology is satisfiable. (There is truth value $v(\top) = 1$.)

d) A contradiction is unsatisfiable. ($v(\bot) = 0$ for every truth value v.)

Exercise 1.7. Prove the validity of the following for any formula of PL A (see Theorem 1.7).

a) $A \wedge \top \boxminus A$

b) $A \vee \top \boxminus \top$

c) $A \wedge \bot \boxminus \bot$

d) $A \vee \bot \boxminus A$

Exercise 1.8. Determine whether the formulas listed below are satisfiable, tautologies, or contradictions. (Try to find the answer using your knowledge of logical connectives first, then prove it formally.)

a) $(P \wedge Q) \Rightarrow P$

b) $P \wedge \neg P$

c) $\neg(P \wedge \neg P)$

d) $(P \Rightarrow R) \wedge (P \wedge \neg R)$

e) $P \Leftrightarrow \neg P$

f) $\neg R \vee R$

g) $(P \wedge R) \Leftrightarrow (P \Leftrightarrow R)$

h) $P \Rightarrow \neg P$

i) $P \Rightarrow (Q \Rightarrow (R \Rightarrow P))$

Exercise 1.9. Proposition. A number is divisible by 6 if and only if it is a multiple of 2 and 3. Express the property *A number is a multiple of 2 and 3.* as

1. a necessary condition for divisibility by 6,
2. a sufficient condition for divisibility by 6.

Identify elementary propositions within the original proposition and formalize it using letters of your choice.
Which of the below are true?

a) divisibility by 2 is a necessary condition for divisibility by 6,

b) divisibility by 2 is a sufficient condition for divisibility by 6,

c) divisibility by 6 is a necessary condition for divisibility by 3,

d) divisibility by 6 is a sufficient condition for divisibility by 3,

e) if a number is not divisible by 6 then it is not divisible by 2 and 3,

f) if a number is not divisible by 6 then it is not divisible by 2 or 3,

g) if a number is not divisible by 6 then it is not divisible by 2,

h) if a number is divisible by 2 then it is divisible by 2 or 3,

i) if a number is divisible by 2 or 3 then it is divisible by 2,

j) if a number is divisible by 2 and 3 then it is divisible by 2

1.3 More exercises

Exercise 1.10. List all subformulas and construct truth tables for the following formulas. Decide whether they are satisfiable, contradictions or tautologies.

a) $(A \Rightarrow B) \Leftrightarrow (B \Rightarrow A)$ (! common mistake!)

b) $(\neg A \Rightarrow \neg B) \Leftrightarrow (A \Rightarrow B)$ (! common mistake!)

c) $(A \wedge B) \Rightarrow \neg(A \Leftrightarrow \neg B)$

d) $(\neg A \Rightarrow B) \vee \neg(A \Leftrightarrow \neg B)$

e) $(\neg A \Rightarrow B) \vee (C \wedge \neg B)$

f) $(\neg A \wedge B) \wedge (C \wedge \neg B)$

g) $\neg(A \vee B) \Rightarrow (A \wedge \neg B)$

h) $(A \Rightarrow B) \Leftrightarrow ((\neg A \wedge C) \vee \neg B)$

Exercise 1.11. Prove the validity of the logical laws in Theorem 1.8 (i.e., prove every single logical equivalence there to be a tautology)

Exercise 1.12. * The formula

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

contains only implications. Is it a tautology?

Exercise 1.13. Let's have two elementary propositions:

- A number x is divisible by 4 (denote by A), and
- A number x is divisible by 2 (denote by B).

Formulate which is a necessary condition for which and express as a formula of PL.

Exercise 1.14. * In the Island of Knights and Knaves, knights always make true statements and knaves always make false ones. Determine of which type are the natives.

a) We meet two natives, A and B, and A says: "At least one of us is a knave."

b) We meet two natives, A and B and A says: "Either I am a knave or B is a knight."

c) We meet two natives, A and B, and A says: "I am a knave but B isn't."

d) We meet three of them, A, B and C, and A and B make the following statements:

   - A: "All of us are knaves.".
   - B: "Exactly one of us is a knight."

e) * Once when I visited the island of knights and knaves, I came across two of the inhabitants resting under a tree. I asked one of them, "Is either of you a knight? "He responded, and I knew the answer to my question. To which type of person I addressed the question? Is he a knight or a knave? And what is the other one? I can assure you, I have given you enough information to solve this problem.