BIE-DML - Discrete Mathematics and Logic

Tutorial 6

Relations, binary relations, equivalence and order relations

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6.1 Introduction

6.1.1 Relations and binary relations

We begin by revising the basic notions related to the topic of binary relations. As mentioned in the previous Tutorial, binary relations are a generalization of mappings.

Definition 6.1. Consider two sets X and Y in a universe $\mathcal{U}$. Let R be an arbitrary subset of $X \times Y$. We call the triple (R, X, Y) a **binary relation** from X to Y. The set X is called the **left domain** and the set Y the **right domain** of the relation (R, X, Y). A binary relation (R, X, X) is called a binary relation **on** X. We very often drop the word "binary" and just say "a relation" instead. Furthermore, if the domains of (R, X, Y) are evident from the context, we simply write R

- We usually write aRb to express the fact that $(a, b) \in R$ and we say "an element $a \in X$ is related to an element $b \in Y$ by the relation R", or, "a is related to b by R".
- The domain of a relation R is the set $\mathfrak{D}(R) = \{x \in X : \exists\, y \in Y,\ xRy\} \subseteq X$.
- The range of a relation R is the set $\mathfrak{Im}(R) = \{y \in Y : \exists\, x \in X,\ xRy\} \subseteq Y$.
- There are three special types of relations on a set X:
  - the empty relation $O_X = \emptyset$,
  - the diagonal (identity) relation $\Delta_X = \{(x, x) : x \in X\}$, and
  - the total (complete) relation $X \times X$.
- The inverse relation of a relation (R, X, Y) is the relation (R^{-1}, Y, X) such that

$$R^{-1} = \{(y, x) : (x, y) \in R\} \subseteq Y \times X.$$

- Given two binary relations (S, X, Y) and (R, Y, Z), the composition of R and S is the binary relation $(R \circ S, X, Z)$ defined as

$$x(R \circ S)z \Leftrightarrow \exists\, y \in Y :\ xSy \wedge yRz.$$

- The composition of relations is associative, i.e., $(R \circ S) \circ T = R \circ (S \circ T)$. Moreover, if $R \subseteq X \times X$ then we can define the powers of the relation R as

$$R^n = \begin{cases} R \circ (R^{n-1}) = (R^{n-1}) \circ R = \underbrace{R \circ R \circ \cdots \circ R}_{n\text{-times}} & \text{for } n > 0, \\ \Delta_X & \text{for } n = 0, \\ (R^{-1})^{-n} = (R^{-n})^{-1} & \text{for } n < 0. \end{cases}$$

- Given a relation $R \subseteq X \times X$, we define the transitive closure R^+, and the reflexive-transitive closure R^*, as follows:

$$R^+ = \bigcup_{i=1}^{\infty} R^i = R \cup R^2 \cup \cdots = \{(a, b) \in X \times X : aRb \vee aR^2b \vee \dots\},$$

and

$$R^* = \Delta_X \cup R^+ = \bigcup_{i=0}^{\infty} R^i = \Delta_X \cup R \cup R^2 \cup \cdots = \{(a, b) \in X \times X : a = b \vee aRb \vee aR^2b \vee \dots\}.$$

Note that if X is finite then there is a number $n \in \mathbb{N}$ such that

$$\bigcup_{i=1}^{n+1} R^i = \bigcup_{i=1}^{n} R^i$$

i.e., the closures can be computed in a finite number of steps.

Definition 6.2. We study a few special properties of binary relations on a set X. Assume that $R \subseteq X \times X$. We say that R is

$$\begin{array}{llcll}
(RE) & reflexive & \Leftrightarrow & (\forall x \in X : xRx) & \Leftrightarrow \; \Delta_X \subseteq R, \\
(SY) & symmetric & \Leftrightarrow & (\forall x, y \in X : xRy \Rightarrow yRx) & \Leftrightarrow \; R^{-1} = R, \\
(TR) & transitive & \Leftrightarrow & (\forall x, y, z \in X : xRy \wedge yRz \Rightarrow xRz) & \Leftrightarrow \; R^2 \subseteq R, \\
(AN/ANS) & antisymmetric & \Leftrightarrow & (\forall x, y \in X : xRy \wedge yRx \Rightarrow x = y) & \Leftrightarrow \; R \cap R^{-1} \subseteq \Delta_X, \\
(AS) & asymmetric & \Leftrightarrow & (\forall x, y \in X : xRy \Rightarrow \neg(yRx)) & \Leftrightarrow \; R \cap R^{-1} = O_X, \\
(IR) & irreflexive & \Leftrightarrow & (\forall x \in X : \neg(xRx)) & \Leftrightarrow \; R \cap \Delta_X = O_X.
\end{array}$$

Remark 6.3. Note that we do not use the notions related to mappings as "image" or "preimage". There is no unique interpretation of these notions in n-ary relation, and in case of binary relations we can only say two elements are or are not in the relation (not "from-to" relation). This is sometimes confusing if you try to interpret relation in natural language: "a and b are related by R" could mean aRb as well as bRa. So, can we say $aRb \Leftrightarrow bRa$? No, we have to be careful, it does not hold in all relations!

We can represente and visualize (binary) relations as follows.

1. **By logical formulas:** Very useful representation of relations, and, in case of infinite relations, the only way. Consider

$$R = \{(a, b) \in X \times X : \text{"statement about } a, b\text{"},$$

or

$$aRb \Leftrightarrow \text{"statement about } a, b\text{"}.$$

Then, symmetry is represented by a logical formula

$$\text{"}\forall\ a, b \in X : \text{statement about } a, b \Rightarrow \text{statement about } b, a\text{"}.$$

For example, we can find powers or inversion of a relation as follows: Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be defined as

$$aRb \Leftrightarrow b \geq 2a-$$

Then R^2 is defined by this formula:

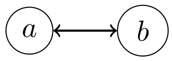
$$a(R^2)b \ \Leftrightarrow\ a(R \circ R)b \ \Leftrightarrow\ \exists\ x \in \mathbb{Z} : aRx \wedge xRb \ \Leftrightarrow\ \exists\ x\ \in \mathbb{Z} : (x \geq 2a) \wedge (b \geq 2x) \ \Leftrightarrow\ b \geq 4a.$$

Thus, the relation R^2 is defined by

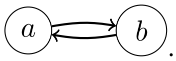
$$R^2 = \{(a, b) \in \mathbb{Z}^2 : 4a \leq b\}.$$

2. **List of pairs:** Finite relations can be directly listed (as a set of pairs from $X \times X$), e.g., $R = \{(1, 1), (1, 2), (2, 3)\} \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$. Listing is not convenient for big sets, because testing the properties means checking all the pairs of the whole set. E.g., to test transitivity, we need to check for all triples $a, b, c \in X$, for which aRb and bRc is true whether also aRc is true. Similarly for power: for each element $a \in X$ we need to find all pairing $b \in X$ satisfying aRb and then for every such b finding c s.t. bRc. Each such chain gives one pair (a, c) to R^2.

3. Diagram: Diagram of a finite relation (R, X, X) is a so-called digraph $G = (V, E)$ (from graph theory) with vertex set $V = X$ and edges $E = R \subseteq V \times V$ with self-loops (pairs (v, v) in the relation). Moreover, if both aRb and bRa, we simply draw



instead of



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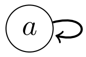
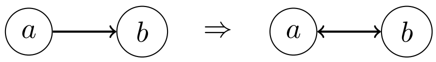
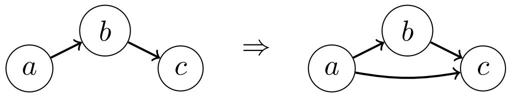
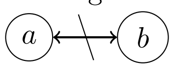
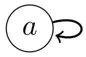
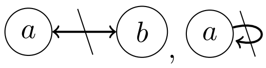
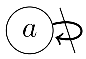
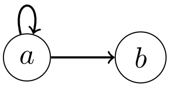
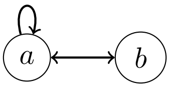
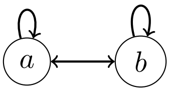
We can test the properties visually as the following table presents:

relation R	diagram of the relation (X, R)
is reflexive (RE)	every vertex $a \in X$ has a self-loop, 
is symmetric (SY)	every edge has arrows on both ends, 
is transitive (TR)	every path of length 2 has an edge as a shortcut, 
is antisymmetric (AN)	no edge of this type in the diagram  except self-loops 
is asymmetric (AS)	no edge of this type in the diagram 
is irreflexive (IR)	no self-loops 

Remark 6.4. Moreover, we need to consider these situations when checking transitivity:

-  is (TR) since $(a, a) \in R \wedge (a, b) \in R \Rightarrow (a, b) \in R$.
-  is not (TR) since $(b, a) \in R \wedge (a, b) \in R$ but (b, b) is not in R.
-  is (TR) since the previous case cannot occur.

3. Matrix: Again, consider a relation $R \subseteq X \times X$ on a finite set $X = \{1, 2, \ldots, n\}$ (we can assign numbers to each element of X). Then the **relation matrix** (or matrix) of R is a square matrix $n \times n$:

$$(M_R)_{i,j} = \begin{cases} 1 & \text{if } iRj, \\ 0 & \text{if } \neg(iRj). \end{cases}$$

The Cartesian representation is similar. Assign the – potentially real – numbers to the elements of X. Then the pair in relation $(a, b) \in R$ represents a point in the plane $(x, y) \in \mathbb{R}^2$.

Note that a matrix can be defined as

$$(M_R)_{i,j} = \text{ number of paths of length 1 from } i \text{ to } j \text{ in the relation diagram}$$

which can be extended to the powers (by inductive definition) $(M_R)^k_{i,j}$:

$$(M_R)^k_{i,j} = \text{ number of paths of length } k \text{ from } i \text{ to } j \text{ in the relation diagram.}$$

We can use this to find the powers of finite set relations:

$$(M_{R^k})_{i,j} = \begin{cases} 1 & \text{if } ((M_R)^k)_{i,j} \geq 1, \\ 0 & \text{if } ((M_R)^k)_{i,j} = 0. \end{cases}$$

The matrix of an inverse relation can be defined as $M_{R^{-1}} = (M_R)^T$.

We can observe the following matrix relationship-properties:

relation R	relation matrix M_R
is reflexive (RE)	$\forall\, i \in \{1, 2, \ldots, n\}: \quad (M_R)_{i,i} = 1$
is symmetric (SY)	M_R is a symmetric matrix, i.e.,. $(M_R)^T = M_R$
is transitive (TR)	$\forall\, i, j \in \{1, 2, \ldots, n\}: \quad (M_{R^2})_{i,j} = 1 \Rightarrow (M_R)_{i,j} = 1$
is antisymmetric (AN)	$\forall\, i, j \in \{1, 2, \ldots, n\},\ i \neq j: \quad \neg\Big((M_R)_{i,j} = 1 \wedge (M_R)_{j,i} = 1\Big)$
is asymmetric (AS)	$\forall\, i, j \in \{1, 2, \ldots, n\}: \quad \neg\Big((M_R)_{i,j} = 1 \wedge (M_R)_{j,i} = 1\Big)$
is irreflexive (IR)	$\forall\, i \in \{1, 2, \ldots, n\}: \quad (M_R)_{i,i} = 0$

Remark 6.5. At the end of this section, we present some properties of composition of relations. Consider three binary relations R, S, T on a set X. Then

- $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$,
- $(R \cap S) \circ T \subseteq (R \circ T) \cap (S \circ T)$,
- $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$,
- $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$,
- $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$,
- $R \subseteq S \Rightarrow (R \circ T) \subseteq (S \circ T)$.

Remark 6.6. Let us introduce the most common student mistakes here. First, if a relation R is defined on a set X, then it is a subset of $X \times X$ ($R \subseteq X \times X = X^2$) and not a subset of X.

Relations can be defined on any pair of sets. Consider the following examples (where X is some set).

- a relation R on the power set $\mathcal{P}(X)$, i.e., $R \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$, where the pairs are not elements of X, but subsets of X that are related; R consists of **pairs of sets**!
- a relation R on the set X^n for an arbitrary $n \in \mathbb{N}$, i.e., $R \subseteq X^n \times X^n$, where the pairs are not elements of the set X, but n-tuples; R consists of **pairs of n-tuples**!

6.1.2 Equivalence and Order Relations

Definition 6.7. Let X be a nonempty set.

- A binary relation R on the set X is an **equivalence** if it is **reflexive, symmetric and transitive**, i.e.,

$$(\Delta_X \subseteq R) \wedge (R = R^{-1}) \wedge (R^2 \subseteq R)\,.$$

- A system of nonempty sets $\mathcal{X} = \{X_i \subseteq X : i \in I\}$ is a **partition set** on X, if the sets in this system are mutually disjoint (i.e., $X_i \cap X_j = \emptyset$ for every $i, j \in I$, $i \neq j$) and cover the set X (i.e., $\bigcup_{i \in I} X_i = X$). Elements X_i of this system are called **partition classes** of set X.

- Let R be an equivalence on X and $a \in X$ an arbitrary element. Then the set

$$[a]_R = \{x \in X : xRa\}$$

  is called an **equivalence class** of R represented by a (element class). The equivalence class can be represented by any of its element, i.e., $aRb \Leftrightarrow [a]_R = [b]_R$.

Theorem 6.8. *Let X be a nonempty set.*

1. *Every equivalence R on X defines exactly one partition set.*
   Consider a system of different classes $[a]_R$ for $a \in X$. This system is called a ***factor set*** *X w.r.t. R and it is denoted by*

$$X/R = \{[a]_R : a \in X\}.$$

2. *Every partition set on X defines exactly one equivalence.*
   Denote by $\{X_i : i \in I\}$ the sets in the partition system on X. Then the relation R defined as

$$\forall\ a, b \in X : aRb \Leftrightarrow (\exists\ j \in I : a \in X_j \wedge b \in X_j)$$

   is an equivalence.

Remark 6.9. There is one notation – $[a]_R$ – but several names. Do not be confused if these sets are called a class of element a, partition set of a system or a class of the equivalence. All of them mean the same.

Definition 6.10. Let X be a nonempty set.

- A binary relation R on the set X is a **(partial) order** if R is **reflexive, antisymmetric and transitive**, i.e.,

$$(\Delta_X \subseteq R) \wedge (R \cap R^{-1} \subseteq \Delta_X) \wedge (R^2 \subseteq R)\,.$$

- If R is a partial order on X then the pair (X, R) is a partially ordered set (**poset**). A relation R on poset (X, R) is often denoted as $\preceq$.

- Let $\preceq$ be a partial order on X. We can extract a sub-relation called a strict order $\prec$ defined as $\prec\ =\ (\preceq \setminus \Delta_X)$ satisfying $x \prec y \Leftrightarrow (x \preceq y \wedge x \neq y)$ for every $x, y \in X$.

- Let R be any binary relation on X. A (transitive) reduction of R, denoted by R_r, is defined as $R_r = R \setminus R^2$. If R is a strict order then the relation R_r is called a covering relation (or the immediate predecessor relation).

Theorem 6.11. *Let X be a nonempty set and let R be a partial order on X (relation RE, TR, AN). R can be described uniquely by its strict relation $R \setminus \Delta_X$ (relation IR,TR,AS), or by the covering relation $(R \setminus \Delta_X)_r = (R \setminus \Delta_X) \setminus (R \setminus \Delta_X)^2$ (relation IR,AS):*

$$\text{partial order} \;\underset{(\cdot)\cup\Delta_X}{\overset{(\cdot)\setminus\Delta_X}{\rightleftarrows}}\; \text{strict order} \;\underset{(\cdot)^+=(\cdot)\cup(\cdot)^2\cup(\cdot)^3\cup\ldots}{\overset{(\cdot)_r=(\cdot)\setminus(\cdot)^2}{\rightleftarrows}}\; \text{covering relation}$$

Definition 6.12. Let R be a partial order on $X \neq \emptyset$. A Hasse diagram of the partial order R is a digraph obtained from the digraph of R as $(R \setminus \Delta_X)_r$. We replace each arc $a \to b$ by an undirected one and place the vertices so that a is below b: $\overset{b}{\underset{a}{|}}$ (see also Figure 6.1).

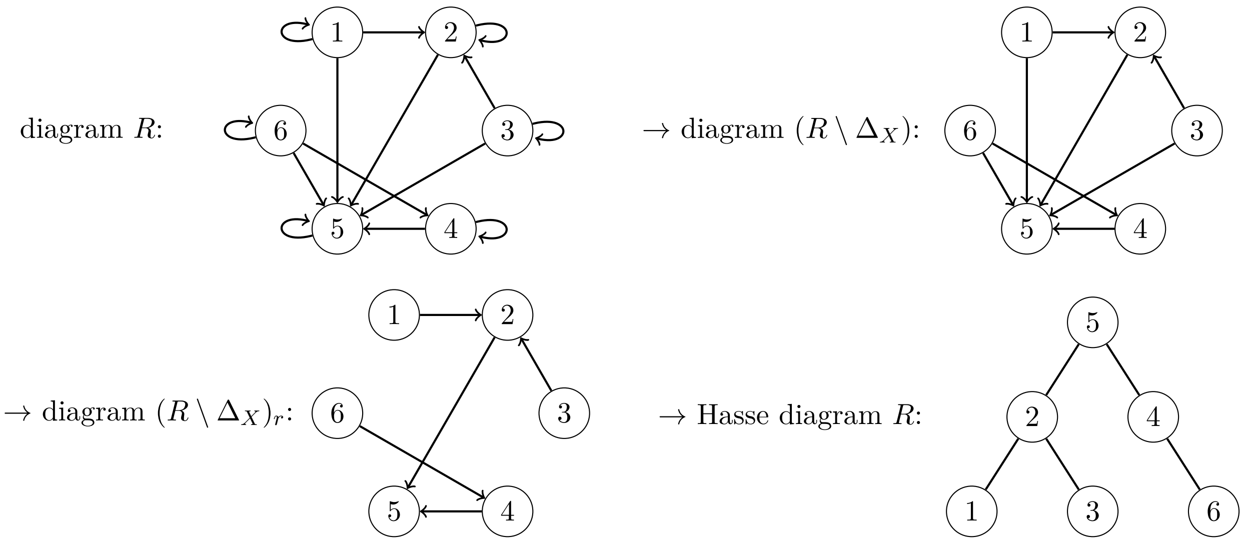


Figure 6.1: Hasse diagram obtained from R on the set $\{1, 2, 3, 4, 5, 6\}$ given by the relation diagram (top left).

Definition 6.13. Let R be a partial order on $X \neq \emptyset$ and let $M \subseteq X$ be an arbitrary nonempty subset of X.

- Two elements $a, b \in X$ are comparable if aRb or bRa; otherwise they are incomparable.
- R is a total order (or a linear order) on X if every pair of elements in X is comparable.
- An element $a \in M$ is a least element (minimum) of M if aRx for every $x \in M$.
- An element $a \in M$ is a greatest element (maximum) of M if xRa for every $x \in M$.
- An element $a \in M$ is a minimal element of M if there is no $x \in M$, $x \neq a$ such that xRa.
- An element $a \in M$ is a maximal element of M if there is no $x \in M$, $x \neq a$ such that aRx.

Remark 6.14. There are several observations regarding these special elements:

- The least and the greatest element are comparable with all elements in the set M.
- Minimal and maximal elements always exist if the set M is finite and these elements need not to be unique in M. They can also can be incomparable with some elements of M.
- If a set has one minimal (resp. maximal) element then this is also the least (greatest resp.) element. Otherwise the least (resp. greatest) element does not exist.

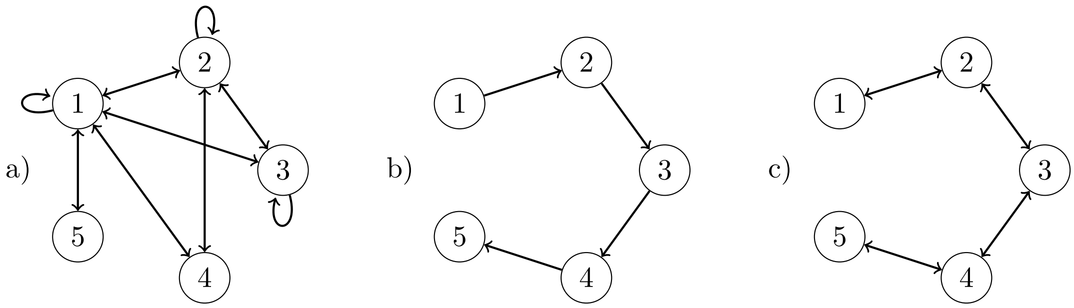
Reference:

1. exercises BIE-ZDM at FIT:
   https://courses.fit.cvut.cz/BIE-ZDM/
2. Wells, Ch.: Discrete Mathematics.

6.2 Exercises

6.2.1 Relations and binary relations

Exercise 6.1. Consider the following relations on the set $X = \{1, 2, 3, 4, 5\}$ given by the graphs below.



For each relation R:

i) find the domain $\mathfrak{D}(R)$ and range $\mathfrak{Im}(R)$:

ii) describe the relation by listing and by matrix;

iii) decide which properties between RE, SY, TR, AN, AS, IR are satisfied;

iv) find the inverse relation R^{-1} and powers R^2, R^3.

Exercise 6.2. Consider the following relations on the set of positive integers $\mathbb{N}$. For each of them find the domain $\mathfrak{D}(R)$ and the range $\mathfrak{Im}(R)$, decide which properties between RE, SY, TR, AN, AS, IR are true.

a) $mRn \Rightarrow m + n$ is even,

b) $mRn \Rightarrow m - n$ is even,

c) $mRn \Rightarrow m + n$ is odd,

d) $mRn \Rightarrow m + n \leq 100$,

e) $mRn \Rightarrow |m - n| \leq 1$,

f) $mRn \Rightarrow n = k \cdot m$ for any $k \in \mathbb{N} \setminus \{1, n\}$,

g) $mRn \Rightarrow m/n = 2^k$ for some $k \in \mathbb{Z}$,

h) $mRn \Rightarrow (m^2 - n^2)$ is divisible by 3.

Exercise 6.3. Let us denote the relations on the set $\{1, 2, 3, 4, 5\}$ in Exercise 6.1 as follows:

$$mRn \Leftrightarrow m + n \leq 6 \qquad mSn \Leftrightarrow m = n - 1\,, \qquad mTn \Leftrightarrow |m - n| = 1\,.$$

Let us consider the following compositions of relations: $S \circ R$, $R \circ S$, $T \circ R$, $R \circ T$, $T \circ S$, $S \circ T$. Choose one of the possible representations:

- both relations and their composition are represented by a formula;
- both relations and their composition are represented by a list;
- both relations and their composition are represented by a diagram;
- both relations and their composition are represented by a matrix.

Exercise 6.4. Find the transitive and the reflexive-transitive closures of relations in Exercise 6.2. Describe them as formulas if $R \subseteq \mathbb{N} \times \mathbb{N}$.

Exercise 6.5. Consider $X = \{0,1\}^4$, thus X is a set of binary words of length 4. Consider a relation R on the set X defined by the formula

$$pRq \Leftrightarrow \text{ words } p \text{ and } q \text{ have a common factor of length at least } 2\,.$$

a) Decide if the relation is RE, SY, TR, AN, AS, IR.

b) Find the transitive closure R^+.

Remarks:

- $X = \{0000, 0001, 0010, \ldots, 1000, 0011, 0101, 0110, 1001, 1010, 1100, 0111, \ldots 1111\}$
- a factor of two strings is a substring which is contained in both strings. Here, factor of length at least two for strings $p = p_1p_2p_3p_4$, $q = q_1q_2q_3q_4$ means: $\exists\ i, j \in \{1, 2, 3\} : p_ip_{i+1} = q_jq_{j+1}$.

Exercise 6.6. Consider the set $X = \{1, 2, 3, 4\}$. Create relations on X (by drawing relation diagrams) so that they satisfy the listed properties (negation symbol $\neg$ here means the relation does not have this property):

a) $(RE) \wedge (SY) \wedge \neg(TR)$

b) $(RE) \wedge (AN) \wedge \neg(TR)$

c) $(RE) \wedge \neg(SY) \wedge \neg(TR)$

d) $\neg(RE) \wedge (SY) \wedge \neg(AN) \wedge (TR)$

e) $\neg(RE) \wedge \neg(SY) \wedge (TR)$

Exercise 6.7. Consider two binary relations R and S on a set X. Decide whether the following statements are true or false:

a) R, S are transitive $\Rightarrow$ $R \cup S$ is transitive.

b) R, S are transitive $\Rightarrow$ $R \cap S$ is transitive.

c) R, S are transitive $\Rightarrow$ $R \circ S$ is transitive.

d) R, S are reflexive $\Rightarrow$ $R \cup S$ is reflexive.

e) R, S are reflexive $\Rightarrow$ $R \cap S$ is reflexive.

f) R, S are reflexive $\Rightarrow$ $R \circ S$ is reflexive.

g) R, S are symmetric $\Rightarrow$ $R \cup S$ is symmetric.

h) R, S are symmetric $\Rightarrow$ $R \cap S$ is symmetric.

i) R, S are symmetric $\Rightarrow$ $R \circ S$ is symmetric.

j) R, S are antisymmetric $\Rightarrow$ $R \cup S$ is antisymmetric.

k) R, S are antisymmetric $\Rightarrow$ $R \cap S$ is antisymmetric.

l) R, S are antisymmetric $\Rightarrow$ $R \circ S$ is antisymmetric.

Exercise 6.8. Find the problem with the "proof" of the statement:

Every relation on set X, which is symmetric and transitive, is also reflexive.

"Proof": Consider $x \in X$; using symmetry we get xRy and yRx, thus due to the transitivity also xRx. R is reflexive.

6.2.2 Equivalence and Order Relations

Exercise 6.9. Consider the set $X = \{1, 2, 3, 4, 5\}$ and the following relations. Decide whether or not they are equivalences on X. If yes, find the respective partition sets of X.

a) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)\}$;

b) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (3, 4), (4, 3)\}$;

c) $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$;

d) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (1, 5), (5, 1), (3, 5), (5, 3)\}$;

e) $R = \{(x, y) \in X^2 : 1 \le x \le 5, 1 \le y \le 5\}$;

f) $R = \{(x, y) \in X^2 : x - y \text{ is divisible by } 4\}$;

g) $R = \{(x, y) \in X^2 : x + y \text{ is divisible by } 3\}$;

h) $R = \{(x, y) \in X^2 : 2 - y \text{ is divisible by } x\}$;

Exercise 6.10. Find equivalences on the set $X = \{1, 2, 3, 4\}$ corresponding to the given partition sets. Find the equivalence class for each $a \in X$:

a) $\{\{1, 2\}, \{3, 4\}\}$;

b) $\{\{1\}, \{2\}, \{3, 4\}\}$;

c) $\{\{1\}, \{2\}, \{3\}, \{4\}\}$;

d) $\{\{1, 2, 3\}, \{4\}\}$;

e) $\{\{1, 2, 3, 4\}\}$;

f) $\{\{1\}, \{2, 4\}, \{3\}\}$.

Exercise 6.11. Consider an equivalence R on a nonempty finite set X (with no further specifications).

a) Describe an equivalence R whose factor set X/R is a single-element set.

b) Describe an equivalence R for which $|R| = |X|$ (where $|\cdot|$ denotes set cardinality, i.e., the number of elements in a set).

c) Describe an equivalence R on the set $\{1, 2, 3, 4, 5, 6\}$ whose factor set X/R has exactly 4 elements.

d) Determine how many equivalences there are on the set $X = \{1, 2, 3\}$.

Exercise 6.12. Consider the set $X = \{1, 2, \dots, 10\}$. Let us define the relation R on the set $X \times X$ (i.e., $R \subseteq (X \times X)^2$) as

$$(a, b)R(c, d) \Leftrightarrow a + d = b + c\,.$$

a) Prove that R is an equivalence on $X \times X$.

b) Describe each class of the equivalence by its representative $(a, b) \in X \times X$.

Exercise 6.13. Consider the set $X = \{1, 2, \dots, 10\}$. Let us define a relation S on $X \times X$ (thus S is a subset of $(X \times X)^2$) as

$$(a, b)S(c, d) \Leftrightarrow a \cdot d = b \cdot c\,.$$

a) Prove that S is an equivalence on $X \times X$.

b) Describe each class of the equivalence by its representative $(a, b) \in X \times X$.

c) Describe the relation S by some property (characterization of this relation).

Exercise 6.14. Decide whether or not the following relations are partial orders on the set $\{1, 2, 3, 4\}$. If yes, find the Hasse diagram, least/greatest and minimal/maximal elements.

a) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 3), (3, 4)\}$;

b) $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$;

c) $R = \{(1, 1), (2, 2), (4, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$;

d) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$;

e) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3), (3, 1), (3, 2)\}$;

Exercise 6.15. Decide whether the following relations are partial orders on $\mathbb{Z} \times \mathbb{Z}$.

a) $(u, v)R(x, y) \Leftrightarrow (u \leq x \wedge v = y)$

b) $(u, v)S(x, y) \Leftrightarrow (u \leq x \wedge v < y)$

c) $(u, v)T(x, y) \Leftrightarrow (u \leq x \wedge v \geq y)$

Exercise 6.16. For a natural number $n \in \mathbb{N}$, denote by $c(n)$ the greatest digit in the decimal representation of n, e.g., $c(1472) = 7$. Decide whether the following relations are partial orders on $\mathbb{N}$:

a) $xRy \Leftrightarrow c(x) \leq c(y)$;

b) $xSy \Leftrightarrow c(x) < c(y)$;

c) $xTy \Leftrightarrow (c(x) < c(y) \vee x = y)$.

Exercise 6.17. Decide which matrices are relation matrices of partial orders:

$$a)\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}, \quad b)\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad c)\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}, \quad d)\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}, \quad e)\begin{pmatrix}1 & 1 & 0 & & \\ 0 & 1 & 1\,0 & 0 & 1\end{pmatrix}.$$

Exercise 6.18. For each condition below find a transitive relation R on the set $X = \{1, 2, 3\}$ such that the condition is true (R_r denotes the reduction relation R, i.e., $R_r = R \setminus R^2$, and $O_X = \emptyset$ denotes an empty relation):

a) $R_r = O_X$;

b) $R_r = R \neq O_X$;

c) $O_X \subsetneq R_r \subsetneq R$.

Exercise 6.19. Consider the partial orders R on the set $X = \{a, b, c, d, e\}$ below. Draw the respective Hasse diagrams and discuss all the options to complete the relations to total orders (Δ_X denotes the diagonal relation on X and R^+ denotes the transitive closure of R):

a) $R = \Delta_X$;

b) $R = \Delta_X \cup \{(a,e),(b,e),(c,e),(d,e)\}$;

c) $R = (\Delta_X \cup \{(a,d),(b,d),(c,d),(d,e)\})^+$;

d) $R = (\Delta_X \cup \{(a,c),(b,c),(c,d),(c,e)\})^+$;

e) $R = (\Delta_X \cup \{(a,b),(a,c),(b,d),(b,e)\})^+$.

Exercise 6.20. Determine the least/greatest, the minimal/maximal elements in the orders in Exercise 6.19 if they exist.

6.3 More exercises

Exercise 6.21. Answer and prove the following (consider a relation R on a nonempty set X):

a) Is there any relation which is both reflexive and irreflexive? If yes, what are the rules.

b) Is there any relation which is both non reflexive and non irreflexive? If yes, what are the rules.

c) If a relation is not reflexive, what can be said about irreflexivity (and vice versa)?

Exercise 6.22 (*)**.** Consider a relation R on a nonempty set X. Define two relations S and T on X^2 as follows (assume $a, b, c, d \in X$):

$$(a,b)S(c,d) \Leftrightarrow aRc \wedge bRd\,,$$
$$(a,b)T(c,d) \Leftrightarrow aRc \vee bRd\,.$$

Discuss for every property RE, SY, TR, AN, AS, IR: "if R has this property then S and/or T have it too"? Prove it or find a counterexample.

Exercise 6.23 (*)**.** Consider a relation R on a nonempty set X. Define two relations S and T on the set $\mathcal{P}(X)$ as follows (assume $A, B \in \mathcal{P}(X)$):

$$(A,B) \in S \Leftrightarrow \forall a \in A\ \forall b \in B : aRb\,,$$
$$(A,B) \in T \Leftrightarrow \exists a \in A\ \exists b \in B : aRb\,.$$

Discuss for every property RE, SY, TR, AN, AS, IR: "if R has this property then S and/or T have it too"? Prove it or find a counterexample.

Exercise 6.24 (**)**.** Let R, S, T as in Exercise 6.23. Discuss the truth/falseness of the following statements for both relations T, S:

- the relation is an equivalence,
- the relation is a partial order,
- the relation is nonempty,
- the relation is total ($X^2 \times X^2$).