BIE-DML - Discrete Mathematics and Logic

# Tutorial 8

Combinatorics

Francesco Dolce, Eva Pernecká, Jitka Rybníčková and Jiřina Scholtzová

Faculty of Information Technology
Czech Technical University in Prague

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## 8.1 Introduction

First we will revise basic combinatorial principles and formulas.

### 8.1.1 Basic Combinatorics

In this section we will always assume that we are selecting $k$ **elements from a set containing** $n$ **distinct elements** if not said otherwise. Recall that $n \geq k$ is not necessary.

- **Permutations**: $k = n$, order matters; the number of permutations with no repetition of $n$-element set is $n!$.
- **Variations without repetition**: elements in the selection cannot be repeated and order matters; the number of variations without repetition is $\frac{n!}{(n-k)!}$.
- **Variations with repetition**: elements in the selection can be repeated and order matters; the number of variations with repetition is $n^k$.
- **Combinations without repetition**: elements in the selection cannot be repeated and order does not matter; the number of combinations without repetition is $\binom{n}{k}$.
- **Combinations with repetition**: elements in the selection can be repeated and order does not matter; the number of combinations with repetition is $\binom{n+k-1}{k}$.

If we are selecting from an $n$-element set which contains $n_1$ identical items of type 1, $n_2$ identical items of type 2, ..., and $n_k$ identical items of type $k$, then the number of all selections (if order matters) of this set is

- **Permutations with repetition**: the number of these permutations is $\frac{n!}{n_1!\,n_2!\ldots n_k!}$.

**Remark 8.1.** Recall first the **binomial coefficient** for non-negative integers $k \leq n$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and some properties of this number:

- $\binom{n}{0} = \binom{n}{n} = 1$,
- $\binom{n}{1} = \binom{n}{n-1} = n$,
- $\binom{n}{k} = \binom{n}{n-k}$,
- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$,
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

There are only a handful of combinatorial problems which can be solved using only the basic formulas above. More often we have to combine them using the following combinatorial principles.

**Addition Principle:** If a selection process can be divided into $N$ separate cases and the process selects exactly one of the possibilities while the $i$-th case can be performed in $n_i$ ways, then the process can be performed in $\displaystyle\sum_{i=1}^{N} n_i$ ways.

**Multiplication Principle:** If a selection process can be divided into $N$ independent phases in such a way that there will be one element selected out of $n_i$ in every phase $i$, then the whole selection can be done in $\prod_{i=1}^{N} n_i$ ways.

**Complement Principle:** Assume that a process can be done in a known number of ways and these ways are decomposed into two groups. If we are able to count the number of ways in the first group, then the second group represents the number of ways calculated as a difference of the total count minus the count of the first group.

### 8.1.2 Advanced Combinatorial Principles

The generalization of the addition principle into the alternative cases, if the alternatives are not (necessarily) disjoint, is called the Principle of inclusion and exclusion.

**Principle of inclusion and exclusion (IN-EX):** Consider finite sets $A_i$ for $i = 1, 2, \dots, n$. Then

$$\Big|\bigcup_{i=1}^{n} A_i\Big| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n-1}\Big|\bigcap_{i=1}^{n} A_i\Big|,$$

where $|A|$ is the cardinality of set $A$.

**Remark 8.2.** The formula uses general summation notion $\sum\limits_{1\le i<j\le n}$, $\sum\limits_{1\le i<j<k\le n}$, and so on, up to $\sum\limits_{1\le i_1<i_2<\dots<i_k\le n}$. Let's review what it means. The first sum is:

$$\begin{array}{rcccccccc}
\sum\limits_{1\le i<j\le n} |A_i \cap A_j| = & |A_1 \cap A_2| + & |A_1 \cap A_3| + & \dots + & |A_1 \cap A_i| + & \dots + & |A_1 \cap A_n| \\
& + & |A_2 \cap A_3| + & \dots + & |A_2 \cap A_i| + & \dots + & |A_2 \cap A_n| \\
& & & & & \vdots & \vdots \\
& & & + & |A_{i-1} \cap A_i| + & \dots + & |A_{i-1} \cap A_n| \\
& & & & & \vdots & \vdots \\
& & & & & + & |A_{n-1} \cap A_n|
\end{array}$$

We can see that all pairs of set intersections $A_i$, $A_j$ such that $1 \le i \le n$, $1 \le j \le n$ and $i < j$ are listed. Similarly the triples $A_i, A_j, A_k$, where indices $i, j, k$ hold $1 \le i \le n$, $1 \le j \le n$, $1 \le k \le n$ and $i < j < k$, are expressed as the sum

$$\sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k|.$$

In general, the sum $\sum\limits_{1\le i_1<i_2<\dots<i_k\le n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$ expresses the sum of intersections of $k$ sets $A_{i_1}, A_{i_2}, \dots, A_{i_k}$, which indices satisfy $1 \le i_1, i_2, \dots, i_k \le n$ and $i_1 < i_2 < \dots < i_k$.

More principles, formulated in terms of putting objects into boxes, follow:

**Dirichlet's principle/Pigeonhole principle:** If we put $n$ objects into $k$ boxes (where $k < n$), then there is a box containing at least two objects.

**Generalized Pigeonhole principle:** If we put $N$ objects into $k$ boxes, then there is a box with at least $\lceil \frac{N}{k} \rceil$ objects.

**Stirling numbers of the 1st kind $s(n,k)$:** In how many ways can $n$ persons be seated around $k$ round tables in such a way that there is at least one person sitting at each table, with additional conditions:

- Two arrangements in which every person has the same neighbor to the left and the same neighbor to the right are considered equal.
- The table order is irrelevant.

The number of these arrangements can be expressed by several recurrent relations:

$$\begin{aligned} s(n,1) &= (n-1)!, \quad \forall n \geq 1, \\ s(n,k) &= 0, \quad \forall k > n, \\ s(n+1,k) &= s(n,k-1) + n \cdot s(n,k), \quad \forall k,n \in \mathbb{N}, 1 \leq k \leq n. \end{aligned}$$

Proof of this recurrent equation is in Exercise 8.17.

**Remark 8.3.** The number $s(n,k)$ expresses the number of permutations according to their number of cycles (counting fixed points as cycles of length one).

**Stirling numbers of the 2nd kind $S(n,k)$:** In how many ways be $n$ objects put into $k$ same boxes if $n \geq k$ and no box is empty. The number of these arrangements is Stirling numbers of the 2nd kind:

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n.$$

The number of these arrangements can be expressed by several recurrent relations:

$$\begin{aligned} S(n,1) &= 1, \quad \forall n \geq 1, \\ S(n,k) &= 0, \quad \forall k > n, \\ S(n+1,k) &= S(n,k-1) + k \cdot S(n,k), \quad \forall k,n \in \mathbb{N}, 1 \leq k \leq n. \end{aligned}$$

Proof of this recurrent equation is in Exercise 8.18.

**Derangements (Malicious cloakroom attendant):** permutation of $n$ distinct objects, where no object is in its right place, is denoted by $D_n$. The number of derangements is

$$D_n = n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

### 8.1.3 Placing objects into boxes

Many combinatorial situations can be translated into the problem of placing objects into boxes. Objects as well as the boxes can be different or identical (in the sense of distinguishable or indistinguishable). In this subsection, we will present the individual combinations of options, including the corresponding formulas. We will also show how some formulas can be derived relatively easily.

| Placing $n$ objects into $k$ boxes | Number of ways |
|---|---|
| distinct objects, distinct boxes, $n_i$ objects in the $i$-th box $n_1 + n_2 + \cdots + n_k = n$ | $\frac{n!}{n_1!\, n_2! \ldots n_k!}$ |
| distinct objects, distinct boxes (possibly empty) | $k^n$ |
| distinct objects, distinct boxes (cannot be empty) | $k! \cdot S(n, k)$ |
| identical objects, distinct boxes (possibly empty) | $\binom{n+k-1}{k-1}$ |
| identical objects, distinct boxes (cannot be empty) | $\binom{n-1}{k-1}$ |
| distinct objects, identical boxes (possibly empty) | $S(n, 1) + S(n, 2) + \cdots + S(n, k)$ |
| distinct objects, identical boxes (cannot be empty) | $S(n, k)$ |
| distinct persons, identical round tables (cannot be empty) | $s(n, k)$ |

- Distinct objects, distinct boxes, $n_i$ objects in the $i$-th box – we create coins with values $1, 2, \ldots, k$ (one value for each box) and denote by $n_i$ the capacity of $i$-th box (i.e., how many coins we can put to $i$-th box). Then placing $n$ distinct objects into $k$ distinct boxes with capacities $n_1, \ldots, n_k$ can be described by values of $n$ coins in the sequence. The number of distinct sequences corresponds to the permutations with repetition of $n$ elements of $k$, which is
$$\frac{n!}{n_1!n_2! \ldots n_k!}.$$
- Distinct objects, distinct boxes (possibly empty) – we select one box from $k$ for each of $n$ objects. Selections are independent therefore the number of ways is $k^n$ (it corresponds to variations with repetition).
- Distinct objects, distinct boxes (cannot be empty) – placing distinct objects into distinct boxes can be represented by a mapping from the set of objects to the set of boxes. This mapping must be surjective if non-empty boxes are required. The number of surjective mappings from a $n$-element set to a $k$-element set, if $n \geq k$, is
$$\sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k - i)^n,$$
which is the definition of Stirling numbers of the second kind and can be expressed as $k! \cdot S(n, k)$.
- Identical objects, distinct boxes (possibly empty) – every placing of $n$ objects to $k$ boxes can be represented by a string of symbols $\bullet$ and $|$, where $\bullet$ is an object and $|$ is the box separator. E.g., $\bullet\bullet \mid \bullet \mid \bullet\bullet$ and $\mid \bullet\bullet\bullet \mid \bullet\bullet$ are two different placings of 5 identical objects into 3 distinct boxes. The first string has two objects in the first box, one in the second box and two in the third box. The second string has no object in the first box, three objects in the second box and two in the third box. Notice that we need only $k - 1$ separators $|$ for $k$ boxes and $n$ objects. The selections are represented by a string of $n + k - 1$ characters. All possible ways of placing the objects into the boxes is then given by selection of $k - 1$ places for separators from $n + k - 1$ positions, i.e., $\binom{n+k-1}{k-1}$.

- Identical objects, distinct boxes (cannot be empty) – similarly to the previous case we place $n$ same objects into $k$ distinct boxes and represent them by a string of characters $\bullet$ and $|$, where $\bullet$ is an object and $|$ is the box separator. The difference here is that $|$ cannot be neither at the beginning nor at the end of the string and no two symbols $|$ are side by side (i.e., there is always at least one symbol $\bullet$ between them). Therefore we take string of $\bullet$ symbols of length $n$ and insert between them separators $|$ except the beginning and end. We have $n-1$ positions for $k-1$ separators which is $\binom{n-1}{k-1}$.
- Distinct objects, identical boxes (cannot be empty) – corresponds to $S(n,k)$, Stirling numbers of the second kind.
- Distinct objects, identical boxes (possibly empty) – using the addition principle, adding over nonempty boxes denoted by $i$, $1 \leq i \leq k$. The number of selections for given $i$ is $S(n,i)$ so we sum up for each $i$, $1 \leq i \leq k$: $S(n,1) + S(n,2) + \cdots + S(n,k)$.
- Distinct persons, identical round tables (cannot be empty) – corresponds to $s(n,k)$, Stirling numbers of the first kind.

### 8.1.4 Probability

A **probability space** is given by a triple $(\Omega, \mathcal{F}, \mathbf{P})$ where

- $\Omega$ is a **sample space**: it is a nonempty set of all outcomes $\omega$ (called **elementary events**) of an experiment;
- $\mathcal{F}$ is a set of **events** and it is a set of subsets of $\Omega$ (i.e., $\mathcal{F} \subseteq \mathcal{P}(\Omega)$): note that events consist of unions of elementary events.
- $\mathbf{P}$ is a **probability** of events: it is a mapping $\mathbf{P} : \mathcal{F} \to \langle 0, 1 \rangle$.

A **classical Probability Space** is a triple $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the following properties:

- $\Omega$ is a **finite, nonempty** set, satisfying $\mathbf{P}(\Omega) = 1$;
- $\mathcal{F}$ is the power set of $\Omega$ ($\mathcal{F} = \mathcal{P}(\Omega)$); and
- the **probability P** of every event $A$ in $(\Omega, \mathcal{F}, \mathbf{P})$ is given by the formula

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|} = \frac{\text{number of favorable outcomes}}{\text{number of all outcomes}}, \quad \forall A \in \mathcal{F};$$

- the **certain event**is the event $\Omega$ and it holds $\mathbf{P}(\Omega) = 1$;
- the **impossible event** is the event $\emptyset$ and it holds $\mathbf{P}(\emptyset) = 0$;
- the **complement event** to an event $A$ in the universe $\Omega$ is denoted as $\overline{A}$ and it holds $\mathbf{P}(\overline{A}) = 1 - \mathbf{P}(A)$;
- given two events $A$ and $B$ we have

$$\begin{aligned} \mathbf{P}(A \cup B) &= \mathbf{P}(A) + \mathbf{P}(B) && \text{if } A, B \text{ are disjoint,} \\ \mathbf{P}(A \cup B) &= \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) && \text{otherwise (principle of IN-EX).} \end{aligned}$$

**Reference:**


1. http://mathforum.org/advanced/robertd/stirling1.html
2. http://www.cs.columbia.edu/ cs4205/files/CM5.pdf
3. http://www.math.cornell.edu/ levine/18.312/alg-comb-lecture-4.pdf

## 8.2 Exercises

It should be noted that the solutions we present here are usually not the only ones possible. Often the same result can be achieved in several different ways. In some cases, therefore, we present several possible procedures to illustrate the point. In some exercises, we will also refer to the principles of placing objects into boxes. The formulas corresponding to these situations are given in the previous section.

**Exercise 8.1.** Consider 1 byte of a memory (8 bits) which is a binary string of length 8.

a) Determine the number of these binary strings.

b) Determine how many strings contain exactly 4 ones.

c) Determine how many strings contain less than 4 ones.

d) * Determine how many strings contain at least two consecutive ones.

**Exercise 8.2.** Determine the probabilities of selected binary string occurrences from Exercise 8.1 *b)* to *d)*.

**Exercise 8.3.** Consider binary strings of length 16 bits (2 bytes). Decide

a) how many of them start with the prefix `0011`;

b) how many of them are palindromes (i.e., $w = w^R$, where $w^R$ is the mirror of $w$);

c) how many of them have `1` in the first or in the last bit;

d) how many of them contain exactly two symbols `1`;

e) how many of them contain at least two symbols `1`;

f) how many of them contain at most eight symbols `1`.

**Exercise 8.4.** Consider strings of length 3 consisting of letters `A, B, C, D, E`. Decide

a) how many distinct strings can be obtained if letter can repeat;

b) how many distinct strings can be obtained with no letter repetition;

c) how many of strings from *a)* start with letter `A`;

d) how many of strings from *b)* start with letter `A`;

e) how many of strings from *b)* do not contain letter `A`;

f) how many of strings from *a)* contain letter `A`;

g) how many of strings from *b)* contain letter `A`.

**Exercise 8.5.** There are 10 different books on the shelf: five about computers, three about math and two about art. In how many ways can these books be ordered respecting the following conditions:

a) all books about computers should be to the left and all art books to the right;

b) all books about computers should be to the left;

c) the books of the same kind form a continuous group;

d) art book will not be placed next to each other.

**Exercise 8.6.** A library department has ten copies of the same book and one copy of ten distinct books. In how many different ways can ten books be selected (we consider distinct copies of the same book to be interchangeable) if:

a) we need one copy of each book;

b) we do allow the possibility of borrowing one book in multiple copies.

**Exercise 8.7.** How many terms we get applying the distributive law as many times as possible onto formula $(x+y)(a+b+c)(e+f+g)(h+i)$?

**Exercise 8.8.** Consider a set $X$ of $2n+1$ elements. Find the number of subsets of $X$ of size at most $n$ (i.e., sets containing at most $n$ elements).

**Exercise 8.9 (*).** Find the number of distinct binary relations $R$ on $n$-element set if:

a) $R$ is reflexive,

b) $R$ is irreflexive,

c) $R$ is symmetric,

d) $R$ is antisymmetric,

e) $R$ is asymmetric.

**Exercise 8.10 (*).** Find the number of total mappings $f$ on $n$-element set $X$ $(n \geq 2)$ if

a) $f : X \to X$, (no restriction),

b) $f : X \to X$, $f$ is injective,

c) $f : X \to X$, $f$ is surjective,

d) $f : X \times X \to X$, (no restriction),

e) $f : X \times X \to X$, $f$ is injective,

f) (*) $f : X \times X \to X$, $f$ is surjective,

g) $f : X \times X \to X$ is a symmetric mapping, i.e., $\forall x, y \in X$ we have $f(x, y) = f(y, x)$.

**Exercise 8.11.** The members of the association of gardeners in Lhotka (Prague 12) are 6 men and 7 women.

a) In how many ways can five members be selected to the association committee?

b) In how many different ways can three men and four women be selected to the committee?

c) In how many different ways can a four-member committee be selected with at least one woman?

d) In how many different ways can a four-member committee be selected with at most one man?

e) In how many different ways can a four-member committee be selected, in which both men and women will be represented by at least one member?

f) In how many different ways can a four-member committee be selected, in which Mr. Smith will be not together with Mrs. Brown (two members who do not work well together)?

**Exercise 8.12.** Consider 8-bit binary strings.

a) How many strings contain exactly three zeros?

b) How many strings contain three consecutive zeros and only ones in the rest of the string?

c) How many strings contain at least two consecutive zeros?

**Exercise 8.13.** Consider a set of 52 French poker cards that have four suits (hearts ♡, diamonds ♢, spades ♠, clubs ♣) and each suit has values **2**, **3**, **4**, **5**, **6**, **7**, **8**, **9**, **10**, **J**, **Q**, **K**, **A**. Each player receives five cards from the same set (called ”a hand”).

a) How many hands have all four aces **A**?

b) How many hands have four cards of the same value?

c) How many hands have heart suit only?

d) How many hands have only two suits?

e) How many hands have cards of all four suits?

f) How many hands have cards with ranks **A**, **2**, **3**, **4**, **5** of the same suit?

g) How many hands have a sequence of cards of the same suit (ace **A** is the lowest)?

h) How many hands have two cards of one suit, two cards of another suit and one card of a third suit?

i) How many hands have Full house (a pair of the same rank and a triple of the same rank).

**Exercise 8.14.** Consider a bag of red marbles, a bag of green marbles and a bag of blue marbles, each of them containing at least 10 marbles.

a) In how many ways can we select 10 marbles?

b) In how many ways can we select 10 marbles containing at least one red marble?

c) In how many ways can we select 10 marbles containing at least one red marble, two green marbles and three blue marbles?

d) In how many ways can we select 10 marbles containing exactly one red marble?

e) In how many ways can we select 10 marbles containing exactly one red marble and at least one blue marble?

f) In how many ways can we select 10 marbles containing at most one red marble?

g) In how many ways can we select 10 marbles containing twice red marbles than the number of green marbles (and some blue)?

**Exercise 8.15.** Determine:

a) how many natural numbers from the interval $\langle 1, 99\rangle$ are divisible by 7;

b) how many natural numbers from the interval $\langle 1, n\rangle$ are divisible by 7;

c) how many natural numbers from the interval $\langle 1, 99\rangle$ are divisible by $k \in \mathbb{N}$;

d) how many natural numbers from the interval $\langle 11, 99\rangle$ are divisible by 7;

e) how many natural numbers from the interval $\langle m, n\rangle$ are divisible by $k$, where $m, n, k \in \mathbb{N}$;

f) how many prime numbers contains the interval $\langle 1, 99\rangle$.

**Exercise 8.16.** We ordered a shipment of fifty memory chips, but four are defective. Suppose each chip has its own number. Find out:

a) in how many different ways can we select four chips from an entire shipment;

b) in how many different ways can we select four working chips;

c) in how many different ways can we select four chips, such that two of them will work;

d) in how many different ways can we select four chips, at least one of them will be defective.

**Exercise 8.17.** Prove the following identities for Stirling numbers of the first kind $s(n, k)$:

a) $s(n, k) = 0, \quad$ pro $k > n$;

b) $s(n, n) = 1, \quad$ for all $n \geq 1$;

c) $s(n, 1) = (n-1)!, \quad$ for all $n \geq 1$;

d) $s(n, n-1) = \binom{n}{2}, \quad$ for all $n \geq 2$;

e) $s(n+1, k) = s(n, k-1) + n \cdot s(n, k), \quad$ for all $n \geq k \geq 1$;

f) $s(n, 2) = (n-1)!(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}), \quad$ for all $n \geq 2$.

**Exercise 8.18.** Prove the following identities for Stirling numbers of the second kind $S(n, k)$:

a) $S(n, k) = 0, \quad$ for $k > n$;

b) $S(n, n) = 1, \quad$ for all $n \geq 1$;

c) $S(n, 1) = 1, \quad$ for all $n \geq 1$;

d) $S(3, 2) = 3$,

e) $S(4, 2) = 7$,

f) $S(4, 3) = 6$,

g) $S(n, 2) = 2^{(n-1)} - 1, \quad$ for all $n \geq 2$;

h) $S(n, n-1) = \binom{n}{2}, \quad$ for all $n \geq 2$;

i) find a formula of $S(n, n-2)$ for $n \geq 4$ and prove the correctness;

j) $S(n+1, k) = S(n, k-1) + k \cdot S(n, k), \quad$ for all $n \geq k \geq 1$;

k) prove that the number of distinct equivalences on $n$-element set is equal $\sum_{k=1}^{n} S(n, k)$.

## 8.3 More exercises

**Exercise 8.19.** We flip the coin ten times in a row and write down the result of each flip (head or tail – `H` or `T`). Results of ten flips are represented as a sequence of `H` and `T`, e.g., `H H H T H T H T H H`. Determine:

a) how many possible results are there;

b) in how many distinct results a head appears exactly three times;

c) in how many distinct results a head appears at most three times;

d) in how many distinct results a head appears in the fifth flip;

e) in how many distinct results we have the same number of heads and tails.

Determine the probability of *b)* to *e)*.

**Exercise 8.20** (*)**.** Determine number of all binary relations on an $n$-element set $X$ which are not total mappings $X \to X$.

**Exercise 8.21** (*)**.** In how many ways can be 10 office desks placed in three offices (distinct if not specified) if:

a) the desks are the same and the capacity of each office is 10 desks;

b) the desks are the same and the capacity of each office is at most 5 desks;

c) the desks are distinct and the capacity of each office is 10 desks;

d) the desks are distinct and the capacity of each office is at most 5 desks;

e) the desks are distinct and first office will have 3 desks, second 2 desks and third 5 desks;

f) the desks are distinct and there will be at least one desk in every office;

g) the desks are distinct and at least 1 and at most 5 desks can be in every office;

h) the desks are distinct, the office are the same, but there will be at least one desk in each one;

i) the desks are distinct and the offices are the same.

**Exercise 8.22** (*)**.** Three children in the kindergarten are playing together with six blue dice. Each child builds a tower of their dice (one of on top of each other).

a) How many different ways can their buildings look like? We distinguish between individual children, not all dice need to be used in the construction, and we also admit the situation that a child (children) has no dice.

b) How many different ways can the towers look like under the same building conditions when they have a total of $n$ cubes available?

c) How many distinct ways will be for $n$ dice and $k$ children (for $n \geq k$) when each child has at least one die?

**Exercise 8.23.** You're preparing the table for Christmas' dinner. You spent all the afternoon crafting special table decorations with names of all guests, so that when they arrive they immediately know where to sit. You'll be a total of 10 people and you have 3 round tables at your disposal in the living room.

a) In how many can you put the name labels if you want to sit everyone around one table, and using the other two for beverages and food?

b) What if you want to only one table is used for bottles of wine, juice, sodas and other drinks, while the other two have people sitting around (and neither of them is empty)?

c) And if you change your mind and decide to use all three tables to sit people?

d) Helas, in every family there are personal grudges and dislikes. How can you put the labels in such a way that Aunt Belle and Uncle Scrooge sit in different tables?

e) What if we want to avoid that a person sits all alone?