

# MPI - Lecture 12

## Smooth optimization

### Optimization methods overview

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Examples of optimization in IT

- Clustering
- Classification
- Model fitting
- Recommender systems
- ...

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Optimization methods

Optimization methods can be:

1. **discrete**, when the support is made of several disconnected pieces (usually finite);
2. **smooth**, when the support is connected (we have a derivative).

They are further distinguished based on how the method calculates a solution:

1. **direct**, a finite number of steps;
2. **iterative**, the solution is the limit of some approximate results;
3. **heuristic**, methods quickly producing a solution that may not be optimal.

Methods are also classified based on randomness:

1. **deterministic**;
2. **stochastic**, e.g., evolution, genetic algorithms, . . . .

## Smooth optimization methods

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Gradient de-  
scend methods

**Goal:** find local minima of  $f : D_f \rightarrow \mathbb{R}$ , with  $D_f \subset \mathbb{R}^n$ .

We assume that  $f$ , its first and second derivatives exist and are continuous on  $D_f$ .

We shall describe an iterative deterministic method from the family of descent methods.

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Descent method  
- general idea

Let  $x^{(1)} \in D_f$ .

We shall construct a sequence  $x^{(k)}$ , with  $k = 1, 2, \dots$ , such that

$$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)},$$

where  $\Delta x^{(k)}$  is a suitable vector (in the direction of the descent) and  $t^{(k)}$  is the length of the so-called **step**.

Our goal is to have  $f(x^{(k+1)}) < f(x^{(k)})$ , except when  $x^{(k)}$  is already a point of local minimum.

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Descent method  
- algorithm  
overview

Let  $x \in D_f$ .

1. Select  $\Delta x$ ;
2. Select  $t > 0$ ;
3. Calculate  $x + t\Delta x$  and store it in  $x$ ;
4. Repeat this loop until the stopping criterion is satisfied.

For instance, a suitable stopping criterion may be the closeness to **0** of the norm of the gradient.

# Numerical Differentiation

## Introduction and motivation

Numerical  
Differentiation -  
Motivation

**Differentiation** is a basic mathematical operation. It is important to have good methods to compute and manipulate derivatives.

Classical methods (real calculus or complex calculus) are of limited value on computers, since most common programming environments do not have support for symbolic computations.

Another complication is the fact that in many practical applications a function is only known *at a few isolated points*. For example, we may measure the position of a car every minute via a GPS (Global Positioning System) unit, and we want to compute its speed. When the position is known at all times (as a mathematical function) we can find the speed by differentiation.



But when the position is only known at isolated times, this is not possible.

The solution is to use approximate methods (i.e., numerical methods) of differentiation.

Main idea

The basic strategy for deriving numerical differentiation methods is to evaluate a function at a few points, find the polynomial that interpolates the function at these points, and use the derivative of this polynomial as an approximation to the derivative of the function.

This technique also allows us to keep track of the **truncation error**, the mathematical error committed by differentiating the polynomial instead of the function itself.

In addition to the truncation error, there are also **round-off** errors, unavoidable when using floating-point numbers to perform calculations with real numbers.

Numerical differentiation is very sensitive to round-off errors, but these errors are quite easy to analyse.

The problem

Let  $f$  be a given function that is known at a number of isolated points.

The problem of **numerical differentiation** consists in computing an approximation to the derivative  $f'$  of  $f$  by suitable combinations of the known function values of  $f$ .

## Newton's difference quotient

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The basic idea  
(1/2)

The standard definition of  $f'(a)$  is by a limit process,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

In the following we will assume that this limit exists, in other words that  $f$  is differentiable at  $x = a$ .

An immediate and natural approximation of  $f'(a)$  is then

$$f'(a) \approx \frac{f(a+h) - f(a)}{h},$$

where  $h$  is a (small) positive number.

This approximation is called the **Newton's (difference) quotient**.

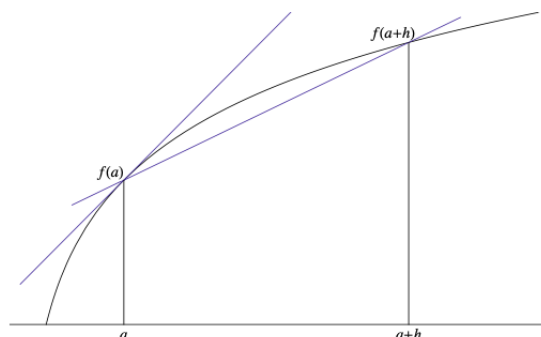
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The basic idea  
(2/2)

The straight line  $p_1$  that interpolates  $f$  at  $a$  and  $a+h$  is given by

$$p_1(x) = f(a) + \frac{f(a+h) - f(a)}{h}(x - a).$$

The derivative of  $p_1(x)$  is exactly the Newton's quotient of  $f$  at  $a$ . Equivalently, the slope of  $p_1$  at  $a$  is an approximation to the slope of  $f$  at  $a$ .




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Left-sided  
version

An alternative approximation is given by the left-sided version:

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}.$$

This approximation behaves similarly, and the analysis is also completely analogous to that of the more common right-sided version.

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Example

Let us consider  $f(x) = \sin x$  at  $a = 0.5$  (using `binary64` floating-point numbers).

We know that the exact derivative is  $f'(x) = \cos x$ , so  $f'(a) \approx 0.8775825619$  with 10 correct digits.

$h$	$\frac{f(a+h) - f(a)}{h}$	$f'(a) - \frac{f(a+h) - f(a)}{h}$
$10^{-1}$	0.8521693479	$2.5 \cdot 10^{-2}$
$10^{-2}$	0.8751708279	$2.4 \cdot 10^{-3}$
$10^{-3}$	0.8773427029	$2.4 \cdot 10^{-4}$
$10^{-4}$	0.8775585892	$2.4 \cdot 10^{-5}$
$10^{-5}$	0.8775801647	$2.4 \cdot 10^{-6}$
$10^{-6}$	0.8775823222	$2.4 \cdot 10^{-7}$

The approximation improves with decreasing  $h$ , as expected.

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Truncation  
error

To analyse errors in numerical differentiation, we use Taylor polynomials with remainders.

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2} f''(\xi_h),$$

where  $\xi_h \in (a, a+h)$ .

Thus

$$f'(a) - \frac{f(a+h) - f(a)}{h} = -\frac{h}{2} f''(\xi_h).$$

This is called the **truncation error** of the approximation.

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Example

Let us consider again  $f(x) = \sin x$  at  $a = 0.5$ . We have  $f''(x) = -\sin x$ , so that the truncation error is

$$f'(a) - \frac{f(a+h) - f(a)}{h} = \frac{h}{2} \sin \xi_h \quad \text{with } \xi_h \in (0.5, 0.5+h).$$

For  $h = 0.1$  the error lies in the interval

$$[0.05 \sin 0.5, 0.05 \sin 0.6] = [2.397 \cdot 10^{-2}, 2.823 \cdot 10^{-2}].$$

As  $h$  became even smaller, the number  $\xi_h$  will approach  $0.5$  and  $\sin \xi_h$  will approach  $\sin 0.5 \approx 0.479426$ . So, for  $h = 10^{-n}$ , the error tends to

$$\frac{10^{-n}}{2} \sin 0.5 \approx 0.2397 \cdot 10^{-n}.$$

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Approximation  
of the trunca-  
tion error

In general, if  $f''(x)$  is continuous, then  $\xi_h$  will approach  $a$  when  $h$  goes to zero. Thus we can approximate  $f''(\xi_h) \approx f''(a)$ .

**Theorem 1.** *The truncation error when using Newton's quotient to approximate  $f'(a)$  is given approximately by*

$$\left| f'(a) - \frac{f(a+h) - f(a)}{h} \right| \approx \frac{h}{2} |f''(a)|.$$

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The exact value of the truncation error is given by

$$\left| f'(a) - \frac{f(a+h) - f(a)}{h} \right| = \frac{h}{2} |f''(\xi_h)| \quad \text{with } \xi_h \in (a, a+h).$$

Upper bound on  
the truncation  
error

**Theorem 2.** *Suppose  $f$  has continuous derivatives up to order two near  $a$ . The truncation error is bounded by*

$$\left| f'(a) - \frac{f(a+h) - f(a)}{h} \right| \leq \frac{h}{2} \max_{x \in [a, a+h]} |f''(x)|.$$

Note that we included the extremal values of the interval  $[a, a+h]$ .

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Round-off error  
- an example

When computing the approximation with small values of  $h$  we have to perform the critical operation  $f(a+h) - f(a)$ , i.e., the subtraction of two almost equal numbers. This may lead to large round-off errors.

Let us consider again  $f(x) = \sin$  at  $a = 0.5$  and the correct value with ten digits is  $f'(0.5) \approx 0.8775825619$ . If we check values of  $h$  smaller than  $10^{-6}$  we find

$h$	$\frac{f(a+h) - f(a)}{h}$	$f'(a) - \frac{f(a+h) - f(a)}{h}$
$10^{-7}$	0.8775825372	$2.5 \cdot 10^{-8}$
$10^{-8}$	0.8775825622	$-2.9 \cdot 10^{-10}$
$10^{-9}$	0.8775825622	$-2.9 \cdot 10^{-10}$
$10^{-11}$	0.8775813409	$1.2 \cdot 10^{-6}$
$10^{-14}$	0.8770761895	$5.1 \cdot 10^{-4}$
$10^{-15}$	0.8881784197	$-1.1 \cdot 10^{-2}$
$10^{-16}$	1.110223025	$-2.3 \cdot 10^{-1}$
$10^{-17}$	0.0000000000	$8.8 \cdot 10^{-1}$

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Round-off error  
in the function  
values

In a previous lecture we saw that the relative error  $\epsilon_1$  in double precision is bounded by

$$|\epsilon_1| = \left| \frac{\text{fl}(f(a)) - f(a)}{f(a)} \right| \leq 5 \cdot 2^{-53} \approx 6 \cdot 10^{-16}.$$

Note that  $\epsilon_1$  depends both on  $a$  and  $f$ .

Let us denote by  $\epsilon^*$  the maximum relative error that occurs when real numbers are represented by floating-point numbers, and there is no underflow or overflow.

Thus

$$\text{fl}(f(a)) = f(a)(1 + \epsilon_1) \quad \text{and} \quad \text{fl}(f(a+h)) = f(a+h)(1 + \epsilon_2),$$

where  $|\epsilon_i| \leq \epsilon^*$  for  $i = 1, 2$ .

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Round-off error  
in the derivative  
(1/2)

The main source of round-off in subtraction is the replacement of the numbers to be subtracted by nearest floating-point numbers. We therefore consider the computed approximation to be

$$\frac{\text{fl}(f(a+h)) - \text{fl}(f(a))}{h},$$



and ignore the error in the division by  $h$ . Hence

$$\begin{aligned} f'(a) - \frac{\text{fl}(f(a+h)) - \text{fl}(f(a))}{h} &= f'(a) - \frac{f(a+h) - f(a)}{h} - \frac{f(a+h)\epsilon_2 - f(a)\epsilon_1}{h} \\ &= -\frac{h}{2}f''(\xi_h) - \frac{f(a+h)\epsilon_2 - f(a)\epsilon_1}{h}, \end{aligned}$$

where  $\xi_h \in (a, a+h)$ .

The **truncation error** is proportional to  $h$ , while the **round-off error** is proportional to  $1/h$ .

When  $h$  is small, we may assume that  $f(a+h) \approx f(a)$ . Thus

$$f'(a) - \frac{\text{fl}(f(a+h)) - \text{fl}(f(a))}{h} \approx -\frac{h}{2}f''(a) - \frac{\epsilon_2 - \epsilon_1}{h}f(a).$$

The most uncertain term is  $\epsilon_2 - \epsilon_1$ . The magnitude of relative errors in **binary64** is about  $10^{-17}$ . If they are of opposite signs, this magnitude may be doubled, so we replace  $\epsilon_2 - \epsilon_1$  by  $2\tilde{\epsilon}(h)$ .

$$f'(a) - \frac{\text{fl}(f(a+h)) - \text{fl}(f(a))}{h} \approx -\frac{h}{2}f''(a) - \frac{2\tilde{\epsilon}(h)}{h}f(a).$$

Hence

$$\tilde{\epsilon}(h) \approx -\frac{h}{2f(a)} \left( f'(a) - \frac{\text{fl}(f(a+h)) - \text{fl}(f(a))}{h} + \frac{h}{2}f''(a) \right)$$

Let us consider again  $f(x) = \sin$  at  $a = 0.5$  and the correct value with ten digits is  $f'(0.5) \approx 0.8775825619$ .

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Round-off error  
in the derivative  
(2/2)

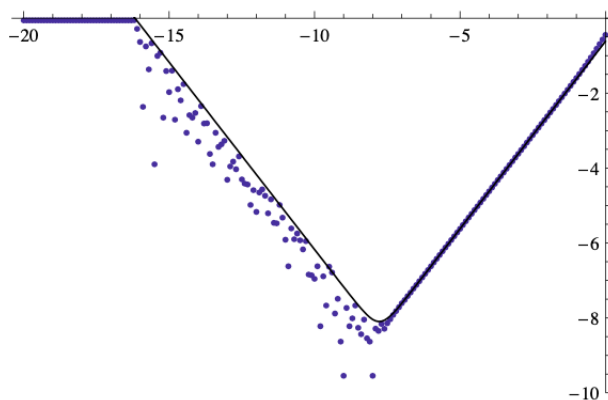
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Round-off error  
in the deriva-  
tive - an exam-  
ple (1/2)

$h$	$\frac{f(a+h) - f(a)}{h}$	$f'(a) - \frac{f(a+h) - f(a)}{h}$	$\bar{\epsilon}(h)$
$10^{-7}$	0.85775825372	$2.5 \cdot 10^{-8}$	$-7.6 \times 10^{-17}$
$10^{-8}$	0.8775825622	$-2.9 \cdot 10^{-10}$	$2.8 \times 10^{-17}$
$10^{-9}$	0.8775825622	$-2.9 \cdot 10^{-10}$ ↓	$5.5 \times 10^{-19}$ ↓
$10^{-11}$	0.8775813409	$1.2 \cdot 10^{-6}$ ↑	$-1.3 \times 10^{-17}$ ↑
$10^{-14}$	0.8770761895	$5.1 \cdot 10^{-4}$	$5.3 \times 10^{-18}$
$10^{-15}$	0.8881784197	$-1.1 \cdot 10^{-2}$	$1.1 \times 10^{-17}$
$10^{-16}$	1.110223025	$-2.3 \cdot 10^{-1}$	$2.4 \times 10^{-17}$
$10^{-17}$	0.0000000000	$-9.2 \cdot 10^{-1}$	$-9.2 \times 10^{-18}$

Round-off error  
in the deriva-  
tive - an exam-  
ple (2/2)

Numerical approximation of the derivative of  $f(x) = \sin x$  at  $x = 0.5$  using Newton's quotient. The plot is a  $\log_{10} - \log_{10}$  plot.



The point  $-10$  on the horizontal axis corresponds to  $h = 10^{-10}$ , and the point  $-6$  on the vertical axis corresponds to an error of  $10^{-6}$ .

Approximation  
of the round-off  
error

The round-off error is given by

$$\begin{aligned}
 \left| f'(a) - \frac{\text{fl}(f(a+h)) - \text{fl}(f(a))}{h} \right| &\approx \left| -\frac{h}{2} f''(a) - \frac{\epsilon_2 - \epsilon_1}{h} f(a) \right| \\
 &\leq \frac{h}{2} |f''(a)| + \frac{|\epsilon_2 - \epsilon_1|}{h} |f(a)| \\
 &\leq \frac{h}{2} |f''(a)| + \frac{|\epsilon_2| + |\epsilon_1|}{h} |f(a)| \\
 &\leq \frac{h}{2} |f''(a)| + \frac{2 \max\{\epsilon_1, \epsilon_2\}}{h} |f(a)|
 \end{aligned}$$

**Theorem 3.** *The round-off error when using Newton's quotient to approximate  $f'(a)$  is roughly bounded by*

$$\left| f'(a) - \frac{f(f(a+h)) - f(f(a))}{h} \right| \lesssim \frac{h}{2} |f''(a)| + \frac{2\epsilon(h)}{h} |f(a)|$$

where  $\epsilon(h) = \max\{\epsilon_1, \epsilon_2\}$ .

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Upper bound on the round-off error

**Theorem 4.** *Suppose  $f$  has continuous derivatives up to order two near  $a$ . The round-off error is (approximately) bounded by*

$$\left| f'(a) - \frac{f(f(a+h)) - f(f(a))}{h} \right| \leq \frac{h}{2} M_1 + \frac{2\epsilon^*}{h} M_2.$$

where  $M_1 = \max_{x \in [a, a+h]} |f''(x)|$  and  $M_2 = \max_{x \in [a, a+h]} |f(x)|$ .

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Optimal choice of  $h$

The previous example shows that there is an optimal value of  $h$  which minimises the total error. If, instead of  $\epsilon_h$  we consider  $\epsilon^*$  we have the error estimate

$$E(a, h) = \frac{h}{2} |f''(a)| + \frac{2\epsilon^*}{h} |f(a)|.$$

To find the value of  $h$  which minimises this expression, let us consider

$$E(a, h)' = \frac{|f''(a)|}{2} - \frac{2\epsilon^*}{h^2} |f(a)|.$$

If we solve the equation  $E(a, h)' = 0$  we obtain the approximate optimal value.

**Theorem 5.** *Suppose  $f$  has continuous derivatives up to order two near  $a$ . The value of  $h$  which minimizes the total error (truncation error + round-off error) is approximately*

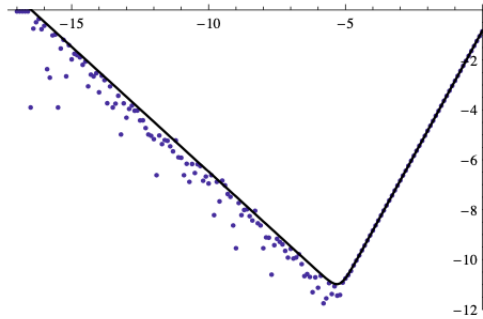
$$h^* \approx 2 \frac{\sqrt{\epsilon^* |f(a)|}}{\sqrt{|f''(a)|}}.$$

- Symmetric version of Newton's quotient

We find an approximation to  $f'(a)$  using the values:  $f(a-h)$ ,  $f(a)$  and  $f(a+h)$ .

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}.$$

$$\text{Error: } \left| f'(a) - \frac{\text{fl}(f(a+h)) - \text{fl}(f(a-h)))}{2h} \right| \lesssim \frac{h^2}{6} |f'''(a)| + \frac{\epsilon^* |f(a)|}{h}.$$



- A four-point differentiation method

We choose as interpolation points  $a-2h$ ,  $a-h$ ,  $a+h$  and  $a+2h$ .

$$f'(a) \approx \frac{f(a-2h) - 8f(a-h) + 8f(a+h) - f(a+2h)}{12h}.$$

$$\left| f'(a) - \frac{\text{fl}(f(a-2h)) - 8\text{fl}(f(a-h)) + 8\text{fl}(f(a+h)) - \text{fl}(f(a+2h))}{12h} \right| \lesssim \frac{h^4}{18} |f^{(4)}(a)| + \frac{3\epsilon^*}{h} |f(a)|.$$

