MPI - Lecture 8

Fuzzy logic

Motivation

Introduction

Consider having a pot of water having temperature of x degrees Celsius.

Is the water **hot**? Is the water **cold**?

Sometimes we want to describe systems by properties which are not evaluated as `true` or `false` (and we do not have the exact value of x).

Fuzzy logic / fuzzy control systems allow such description: truth and falsehood notions are graded and allow to state, for instance, that the water is "**tepid**".

Basic definitions

Universe and crisp sets

Let U denote the universe, that is, our playground containing every set that we may consider.

A set $A \subset U$ can be given by its **characteristic function**:

$$\chi_A : U \to \{0,1\}, \qquad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

There is a bijection between sets and characteristic functions, so we identify each set with its characteristic function.

A is a set in the ordinary sense, sometimes called a crisp set.

Fuzzy sets

Fuzzy sets generalize this concept and allow elements to belong to a given set with a certain *degree*.

We replace the characteristic function by a **membership function**

$$\mu_A : U \to [0,1].$$

A **fuzzy subset** A of a set X is a function $\mu_A : X \to [0,1]$.

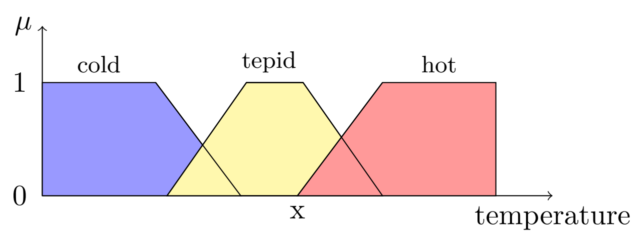
For every element $x \in X$, the **degree of membership** of x to A is given by $\mu_A(x) \in [0,1]$.

Example

Let $X = [0,100]$ be the set of temperatures of water in our pot.

We consider three fuzzy subsets of X to describe cold, tepid and hot temperatures.

The membership functions may be given as follows:



Operations on crisp sets

Given a set X and its power set $\mathcal{P}(X)$ (the set of all subsets of X), the operations of **union**, **intersection**, and **complement** are given as follows (for usual sets):

$$\begin{aligned} A \cup B &= \{x : \ x \in A \text{ or } x \in B\}, \\ A \cap B &= \{x : \ x \in A \text{ and } x \in B\}, \\ A^{\complement} = X \setminus A &= \{x \in X : \ x \notin A\}. \end{aligned}$$

How do these operations translate to characteristic functions?

$$\begin{aligned} \chi_{A \cup B} &= \max\{\chi_A, \chi_B\}, \\ \chi_{A \cap B} &= \min\{\chi_A, \chi_B\}, \\ \chi_{A^{\complement}} &= 1 - \chi_A. \end{aligned}$$

Operations on fuzzy sets

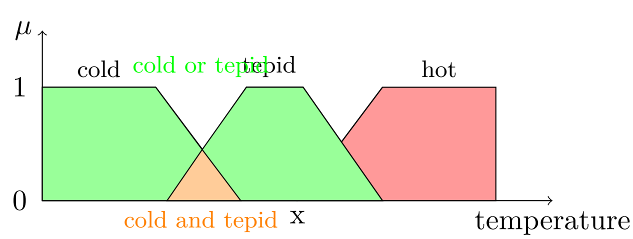
For fuzzy sets, we can define the membership function of a union, intersection, or a complement in the same way.

Let A and B be two *fuzzy* subsets of X.

We set

$$\begin{aligned} \mu_{A \cup B} &= \max\{\mu_A, \mu_B\}, \\ \mu_{A \cap B} &= \min\{\mu_A, \mu_B\}, \\ \mu_{A^{\complement}} &= 1 - \mu_A. \end{aligned}$$

Examples



Operations revisited

Our choice for fuzzy set operation was fast.

Let A and B be two subsets of X. We have

$$\begin{aligned}\chi_{A\cap B}(x) &= \min\{\chi_A(x), \chi_B(x)\}\\ &= \chi_A(x)\cdot\chi_B(x)\\ &= \max\{0,\ \chi_A(x)+\chi_B(x)-1\}\,.\end{aligned}$$

We can extend the second definition to membership functions and obtain another definition of union and intersection of fuzzy sets.

We shall do this in a more general fashion.

t-norms

We have the following requirements of a mapping

$$\star : [0,1]\times[0,1]\to[0,1]$$

that would interpret intersection of two fuzzy sets.

1. $1\star x = x$ for all $x\in[0,1]$,
2. $0\star x = 0$ for all $x\in[0,1]$,
3. $x\star y = y\star x$ for all $x, y\in[0,1]$ (*commutativity*),
4. $(x\star y)\star z = x\star(y\star z)$ for all $x, y, z\in[0,1]$ (*associativity*),
5. $x\le y$ and $w\le z$ implies $x\star w\le y\star z$ (*monotonicity*).

Various t-norms

The following t-norms are usually considered.

Let $x, y\in[0,1]$.

(i) **Gödel** t-norm: $x\star y = \min\{x, y\}$,

(ii) **product** t-norm: $x\star y = x\cdot y$,

(iii) **Łukasiewicz** t-norm: $x\star y = \max\{0,\ x+y-1\}$,

(iv) **Hamacher product** t-norm: $x \star y = \begin{cases} 0 & \text{if } x = y = 0 \\ \dfrac{xy}{x + y - xy} & \text{otherwise} \end{cases}$,

(v) ...

The distinct t-norms give us distinct strategies on how to interpret intersection of fuzzy sets.

If we have intersection and complement, we define union by $A \cup B = \left(A^{\complement} \cap B^{\complement}\right)^{\complement}$ (De Morgan's laws).

Fuzzy control systems

Reasoning in fuzzy logic

In classical logic we can have the following statements:

If "the water is cold" is `true`, then "my shower is bad" is `true`.

An implication is in fact a mapping

$$\{0,1\} \times \{0,1\} \to \{0,1\}.$$

In fuzzy logic, to interpret such implications, we consider "the water is cold" and "my shower is bad" as fuzzy sets and we decide using an **implication** function

$$[0,1] \times [0,1] \to [0,1].$$

This is sometimes called approximate reasoning.

Implication

An **implication** is a function $I : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions for all $x, y, z \in [0,1]$:

1. If $x \leq z$, then $I(x,y) \geq I(z,y)$;
2. if $y \leq z$, then $I(x,y) \leq I(x,z)$;
3. $I(0,y) = 1$;
4. $I(x,1) = x$;
5. $I(1,0) = 0$.

Examples:

(i) Mamdani: $I(x,y) = \min\{x,y\}$ (this fails item 3, but usually in knowledge systems we are not interested in rules where the antecedent part is false),

(ii) Willmott: $I(x,y) = \max\{1-x, \min\{x,y\}\}$,

(iii) ...

Standard fuzzy logic controllers

A controller measures some inputs and gives an output following some rules.

For instance, we have the following set of rules:

r1. If "water is cold", then "shower is bad".

r2. If "water is tepid", then "shower is good".

r3. If "water is hot", then "shower is bad".

The fuzzy sets "shower is bad" and "shower is good" are subsets of $Y = [0, 100]$, measuring how good a shower is.

1. Measure the input variables, i.e., the temperature $x_0 \in X$.
2. Transform the measured values into fuzzy sets: we have fuzzy sets with constant membership functions $\mu_{cold}(x_0)$, $\mu_{tepid}(x_0)$, and $\mu_{hot}(x_0)$.
3. Apply all the rules: we obtain 3 *control* fuzzy sets
   - $\mu_{r_1}(y) = I(\mu_{cold}(x_0), \mu_{bad}(y))$,
   - $\mu_{r_2}(y) = I(\mu_{tepid}(x_0), \mu_{good}(y))$,
   - $\mu_{r_3}(y) = I(\mu_{hot}(x_0), \mu_{bad}(y))$.
4. Aggregate the control fuzzy sets into one fuzzy set C.
5. Defuzzify C to obtain the output value $c \in Y$.

Standard fuzzy logic controllers

For each step, there are many possible choices:

- transformation to fuzzy sets (and application of operators to construct the antecedent of all implications in the rules);
- implication itself;
- aggregation;
- defuzzification.

A usual choice is Gödel t-norm and Mamdani for implication, union for aggregation, and a defuzzification by center of gravity:

$$y_0 = \frac{\int_Y y\mu_C(y)\mathrm{d}y}{\int_Y \mu_C(y)\mathrm{d}y}$$

(or replace by sums if Y is discrete).