Mathematics for Informatics

Numerical Mathematics: power methods (lecture 11 of 12)

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Eigenvalues and eigenvectors

A complex number λ is called an **eigenvalue** of the matric $M \in \mathbb{C}^{n,n}$, whenever there exists a non-zero vector $u \in \mathbb{C}^n$ such that

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The eigenvalues of the matrix M are the roots of the **characteristic polynomial** of the M, that is the polynomial

$$p_M(\lambda) := \det(M - \lambda I).$$

Therefore, each matrix $M \in \mathbb{C}^{n,n}$ has at most n different complex eigenvalues.

Diagonalizability of a matrix

A matrix $M \in \mathbb{C}^{n,n}$ is **diagonalizable** when there exist a diagonal matrix $D \in \mathbb{C}^{n,n}$ and a regular matrix $P \in \mathbb{C}^{n,n}$ such that

$$M = PDP^{-1}.$$

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Remind: $M^k = PD^kP^{-1}$.

Remark:

- The columns of the matrix P are the eigenvectors of M. (These eigenvectors form a basis of $\mathbb{C}^n$.)
- The elements of the diagonal matrix D are the eigenvalues of M (with their multiplicity).

Looking for an eigenvector

Let $M \in \mathbb{C}^{n,n}$. Suppose it is diagonalizable and we can order its eigenvalues as follows

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|.$$

We are looking for the eigenvector of the eigenvalue λ_1, the so-called **dominant eigenvalue**. It is a vector u_1 such that

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In general, the matrix need not be diagonalizable, but the ideas would be more complicated (actually, we only require to have one eigenvalue which is the greatest in absolute value).

Applications

Eigenvalues play an importan role in several applications:

- Classification of conics and quadratic forms (geometry).
- Quantum computation, quantum mechanics, asymptotic behaviour of dynamical systems (physics).
- PCA, or *Principal Component Analysis* (big data).
- Recognition of 2D and 3D objects using spectral methods (AI).
- More practical example: **PageRank** mesures a relative importance of WWW documents by ispecting links between them.
  - Its values is in fact an eigenvector of the dominant eigenvalues of a modified adjacency matrix of these links. This matrix satisfies requirement of our problem.
  - **PageRank** is calculated using **power methods**.

Power method: Introduction and assumptions (1/2)

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Given a matrix $M \in \mathbb{C}^{n,n}$ let us consider a regular matrix $P \in \mathbb{C}^{n,n}$ such that

$$M = PDP^{-1}$$

where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Let also suppose that the values are ordered:

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

Note: We suppose that the dominant eigenvalue λ_1 is not degenerate (i.e., that the correspoinding eigenspace has dimension 1).

Power method: Introduction and assumptions (2/2)

We are looking for an eigenvector associated to the eigenvalue λ_1, that is a non-zero vector u_1 such that

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The **power method** is an **iterative method**. We will construct a sequence $(x_k)_k$ as follows: x_0 is chosen randomly and the next terms are determined by

$$x_k = Mx_{k-1} \quad \text{for } k > 0.$$

Equivalently, we have

$$x_k = M^k x_0 \quad k \in \mathbb{N}_0.$$

Power method principle (1/4)

If M is normal, thus diagonalizable, there exist eigenvectors $\{u_1, u_2, \ldots, u_n\}$, which form a basis of $\mathbb{C}^{n,1}$.

If M is not normal, then we need to complete the set of eigenvectors by a basis of the kernel of M.

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The vector x_0 can be written as $x_0 = \alpha_1 u_1 + \cdots + \alpha_n u_n$.
Suppose that $\alpha_1 \neq 0$.

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Coefficients α_i can be absorbed by the eigenvectors $(u_i' = \alpha_i u_i)$ and we have

$$x_0 = u_1' + \cdots + u_n'.$$

Power method principle (2/4)

The recurrent definition of x_k implies

$$\begin{aligned} x_k &= M^k x_0 \\ &= M^k u_1 + \cdots + M^k u_n \\ &= \lambda_1^k u_1 + \cdots + \lambda_n^k u_n. \end{aligned}$$

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The last equality gives

$$x_k = \lambda_1^k \left(u_1 + \left(\frac{\lambda_2}{\lambda_1} \right)^k u_2 + \cdots + \left(\frac{\lambda_n}{\lambda_1} \right)^k u_n \right).$$

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We rewrite it as

$$x_k = \lambda_1^k \left(u_1 + \varepsilon_k \right).$$

Since for all $j > 1$ we have $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$, then $\lim_{k \to +\infty} \varepsilon_k = 0$.

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We have $\|x_k\| \to +\infty$. Thus we need to control the norm: we may set it to 1 at each step (by *normalizing*, i.e., considering $y_k = \frac{x_k}{\|x_k\|}$).

To have convergence also for the case $\lambda_1 < 0$, we need to pick the right direction for the eigenvector so that it does not oscillate. We may do this by setting the largest entry in absolute value to 1 (and thus use the maximum norm).

The speed of convergence is given by λ_2 since $\|\varepsilon_k\| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$

Power method principe (4/4)

How to find the dominant eigenvalue?

If φ is a linear mapping $\varphi : \mathbb{C}^{n,1} \mapsto \mathbb{C}$ such that $\varphi(u_1) \neq 0$, then

$$\frac{\varphi(x_{k+1})}{\varphi(x_k)} = \frac{\varphi\left(\lambda_1^{k+1}\left(u_1 + \varepsilon_{k+1}\right)\right)}{\varphi\left(\lambda_1^{k}\left(u_1 + \varepsilon_{k}\right)\right)} = \frac{\lambda_1^{k+1}\left(\varphi(u_1) + \varphi(\varepsilon_{k+1})\right)}{\lambda_1^{k}\left(\varphi(u_1) + \varphi(\varepsilon_{k})\right)} \ \rightarrow \ \lambda_1 \quad \text{for } k \rightarrow +\infty.$$

The mapping φ can be set to the mapping defined for all $x \in \mathbb{C}^{n,1}$ as $\varphi(x) = x_{(1)}$ where $x_{(1)}$ is the first component x (if $\varphi(u_1) \neq 0$)).

Power method - demonstration in $\mathbb{R}^{n,n}$

Let us find the dominant eigenvector of $M = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$, which satisfies the conditions of power method.

The exact solution is $u_1 = (1, \sqrt{2}+1) = \frac{1}{\sqrt{2}+1}(\sqrt{2}-1, 1)$, with eigenvalue $\lambda_1 = 3 + \sqrt{2}$.

k	$\widehat{x}_k$	$\|\widehat{x}_k - \widehat{x}_{k-1}\|_\infty$
0	$(1.0, 1.0)$	-
1	$(0.5999999999999998, 1.0)$	0.4
2	$(0.4782608695652174\!6, 1.0)$	0.121739130435
3	$(0.4368932038834951\!7, 1.0)$	0.0413676656817
4	$(0.4223194748358862\!2, 1.0)$	0.0145737290476
5	$(0.417120237506185\!1, 1.0)$	0.0051992373297

In the calculations, the maximum entry in absolute value is set to 1 at each step and the convergence criterion $\|\widehat{x}_k - \widehat{x}_{k-1}\|_\infty < 10^{-2}$.

Power method - demonstration in $\mathbb{C}^{n,n}$ (1/2)

Let us consider the matrix

$$M = \begin{pmatrix} 36408 + 16769i & -5412 - 2481i & 107256 + 49397i & -492 - 214i \\ -10656 - 5164i & 1584 + 762i & -31392 - 15210i & 144 + 66i \\ -12876 - 5954i & 1914 + 881i & -37932 - 17539i & 174 + 76i \\ 4329 - 262i & -643 + 39i & 12753 - 771i & -58 + 6i \end{pmatrix}$$

The eigenvalues are $-2i$, $-i$, $3i/2$ and $3/2$.

Let us fix the accuracy at $\varepsilon = 10^{-6}$. The last 7 iterations of $\lambda_1^{(k)}$ are:

0.0000477588150960872 - 1.99991424541241 *i*
-0.0000479821875446196 - 1.99998019901599 *i*
-0.0000272650944159076 - 2.00002375338328 *i*
0.0000271520045767515 - 2.00002973125038 *i*
0.0000154506695115737 - 1.99997272532314 *i*
-0.0000152424622193764 - 1.99999349337182 *i*

Power method - demonstration in $\mathbb{C}^{n,n}$ (2/2)