Mathematics for Informatics

Algebra: Groups
(lecture 4 of 12)

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Outline

- Introduction and motivation
- Hierarchy of sets with one binary operation
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  - Definitions and elementary properties
  - Cayley table
  - Cayley graph

Searching for hidden similarities...

Let us consider this objects:

- the set $\mathbb{Z}$ of integers with the usual sum;
- the set of matrices $\mathbb{R}^{n,n}$ with the operation of matrix multiplication;
- the set of relations on a set A with the operation of relation composition;
- the set $\{0, 1, 2, 3\}$ with the multiplication $\pmod 4$;
- the set of finite automata with the operation of composition;
- the set of all colors with the operation “mixing”;
- ...

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- A (finite or infinite) **set of objects**.
- A **binary operation** mapping two objects onto (exactly) one object (from the same set of objects).

Generally, we speak about a pair of: a **set** and a **binary operation** on it.

We will (mostly) use one of the following notations: $(M, \cdot)$ (multiplicative notation), $(M, +)$ (additive notation), or $(M, \circ)$ (general notation), where

- $M \neq \emptyset$ is a non-empty set, and
- for binary operation we have $\cdot : M \times M \to M$ (resp. $+ : M \times M \to M$, resp. $\circ : M \times M \to M$).

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If we prove some statement for a general structure $(M, \cdot)$*, where* $\cdot$ *is an associative operation, this statement is proved for all particular structures with an associative binary operation!*

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A proof of this statement is reduced to a proof of associativity of the operation!
We can understand a general structure as a ***parent object****, from which particular structures* ***inherit*** *all its properties (see below).*

Example of "inheritance" (1/4)

On the set of non-zero real numbers we prove the following (trivial) theorem:

Theorem

For all $b, c \in \mathbb{R} \setminus \{0\}$, *the equation* $bx = c$ *has solution* $x = b^{-1}c$.

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What was fundamental for the proof: associativity, existence of (left) inverse element, existence of the neutral element.

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- Is there an inverse matrix for all $A \in M$?
  NO! We have to restrict ourselves to the set of regular matrices M_{reg}.

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We have everything needed to prove the theorem for matrices.

Theorem

For all $B, C \in M_{reg}$, the equation $BX = C$ has solution $X = B^{-1}C$.

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Example of "inheritance" (4/4)

Suppose that we are given a pair $(M, \circ)$ where the associativity law holds, for each element $b \in M$ there exists an inverse element, denoted by b^{-1}, and there exists a neutral element e. We will call such pair a group.

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We have a general theorem.

Theorem

For arbitrary elements b, c of a group $(M, \circ)$, the equation $b \circ x = c$ has solution $x = b^{-1} \circ c$.

Proof.

$$
\begin{aligned}
b \circ x &= c && \text{[multiplication on the left by the inverse element } b^{-1}\text{]} \\
b^{-1} \circ (b \circ x) &= b^{-1} \circ c && \text{[moving brackets due to associativity]} \\
(b^{-1} \circ b) \circ x &= b^{-1} \circ c && \text{[for arbitrary } b \text{ we have } b^{-1} \circ b = e\text{]} \\
e \circ x &= b^{-1} \circ c && \text{[for arbitrary } x \text{ we have } e \circ x = x\text{]} \\
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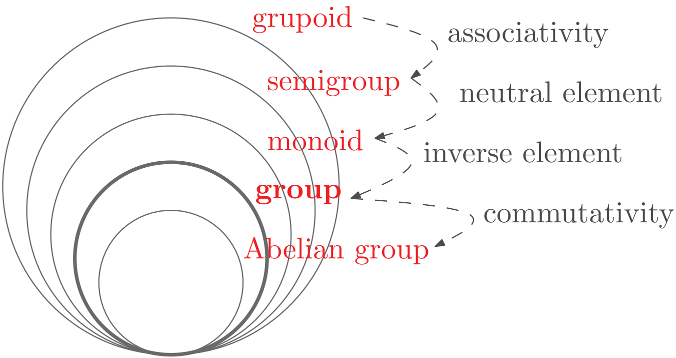
Sets with one binary operation

We call an arbitrary pair "a set and a binary operation" a groupoid. Adding another requirements we get further notions.



Examples

- For the pair $(\mathbb{R} \setminus \{0\}, \cdot)$, the associative and commutative laws hold, the neutral element is 1 and the inverse element for b is $b^{-1} = 1/b$.
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- For the pair $(\mathbb{Z}, +)$ associative and commutative laws hold, the neutral element is 0 and the inverse element for b is $b^{-1} = -b$.
  It is an Abelian group.
- For the pair $(M_{\text{reg}}, \cdot)$ associativity law holds, the neutral element and the inverse exist, but the commutative law is not valid!
  It is a group, but not Abelian.

Mathematical analogy to Object-oriented programming

We can consider the groupoid, monoid, etc., as mathematical (abstract) objects, for which a nonempty set and a binary operation with given properties are defined.

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This analogy could be employed in real programming.

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Definition

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  is called a group.
- *Moreover, if $\circ$ is commutative, we say that a group $(M, \circ)$ is a commutative (or Abelian) group.*

Set closed under the binary operation. What does it mean?

In the definition we require the binary operation $\circ$ to be a "binary operation on M".
This means that the result of a binary operation applied on two elements from M again belongs to M – we say that the **set M is closed under $\circ$.**

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The pair $(\mathbb{Z}_-, \cdot)$ *of negative integers with the usual multiplication is not a groupoid, because it is not closed under the operation:* $(-1) \cdot (-1) = 1 \notin \mathbb{Z}_-$.

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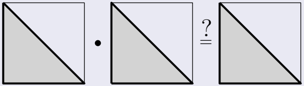
Whether the set is/is not closed under the binary operation is not always obvious.

Example

Let us consider the couple $(M_{triang}, \cdot)$ of lower triangular matrixes with the usual matrix multiplication. Is M_{triang} closed under the operation $\cdot$?



Manual for classification of sets with binary operation

If we have a given pair “a set and a binary operation” and we want to find out whether it is a groupoid, semigroup, monoid, (Abelian) group, we can proceed this way:

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4. Is there an inverse to each element? If `yes`, it is a group; if `not`, END.
5. Does the commutativity law hold? If `yes`, it is an Abelian group; if `not`, END.

Mostly "proofs" in these individual steps are very easy or obvious. Sometimes, they only *seem* obvious.

Groupoid, semigroup, monoid, group – examples (1/4)

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Let us consider the groupoid $(\mathbb{Q}, \circ)$*, where the binary operation* $\circ$ *is defined as the arithmetic mean:*

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Example

Let us consider the groupoid $(\mathbb{Q}, \circ)$*, where the binary operation* $\circ$ *is defined as the arithmetic mean:*

$$a \circ b := \frac{a+b}{2}.$$

Is this structure a semigroup / monoid / group?

In a semigroup, the associative law must hold. Let us claim that for the operation $\circ$ *the law does not hold, and let us prove it by a counterexample:*

$$(2 \circ -2) \circ 4 = 0 \circ 4 = 2 \quad \textit{but} \quad 2 \circ (-2 \circ 4) = 2 \circ 1 = \frac{3}{2}.$$

So, the associative law does not hold, and the structure is not a semigroup. It follows that $\mathbb{Q}$ *with this operation is neither a monoid nor a group.*

Groupoid, semigroup, monoid, group – examples (2/4)

Example

Let us consider a groupoid $(\mathbb{R}^+, \circ)$, *where the binary operation* $\circ$ *is defined as follows:*

$$a \circ b := \frac{a \cdot b}{a + b}.$$

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- *Is* $(\mathbb{R}^+, \circ)$ *a monoid?*

Groupoid, semigroup, monoid, group – examples (3/4)

Example

Let us consider a groupoid $(\mathbb{R}, \cdot)$*, where the binary operation is the usual multiplication of numbers.*

- *Is it a semigroup?*
- *Is it a monoid?*
- *Is it a group?*

Groupoid, semigroup, monoid, group – examples (4/4)

From the definition it follows that each group is a monoid, each monoid is a semigroup and each semigroup is a groupoid. Written in symbols we get:

$$\text{groupoids} \supset \text{semigroups} \supset \text{monoids} \supset \text{groups} .$$

Groupoid, semigroup, monoid, group – examples (4/4)

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From the previous three examples we can be even more specific:

$$\text{groupoids} \supsetneq \text{semigroups} \supsetneq \text{monoids} \supsetneq \text{groups} ,$$

because we have found a groupoid that is not a semigroup, a semigroup that is not a monoid, and a monoid that is not a group.

Uniqueness of neutral element

Theorem

Given a monoid, there exists exactly one neutral element.

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Proof.

Let $(M, \circ)$ be a monoid and e some neutral element (by definition we know that at least one exists!).
We prove *by contradiction* that e is the only neutral element.
By contradiction, assume that in the monoid there exists another neutral element e' different from e.
Using the property of the neutral element, it holds that

$$e' = e' \circ e = e.$$

We get a contradiction with the assumption that $e' \neq e$. □

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Proof.

Let $(G, \circ)$ be a group, a an arbitrary element of the group and a^{-1} one of its inverse elements (from the definition of a group we know that there exists at least one!).

We prove *by contradiction* that a^{-1} is the only one.

Assume that there exists another inverse element $\overline{a}$ different from a^{-1}. Hence it holds that

$$\overline{a} = \overline{a} \circ e = \overline{a} \circ \left(a \circ a^{-1}\right) = (\overline{a} \circ a) \circ a^{-1} = e \circ a^{-1} = a^{-1}$$

where e is the unique neutral element.

Thus we get a contradiction with the assumption that $\overline{a} \neq a^{-1}$. □

Cayley tables for finite groups

If the set M from the pair $(M, \circ)$ has a finite number of elements, its structure (with the given operation $\circ$) could be completely represented by the Cayley table. Its construction is obvious from the following example.

Example

Let us consider $(\mathbb{Z}_4, +_4)$, *i.e., the set of numbers* $\{0, 1, 2, 3\}$ *with addition modulo* 4.

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$+_4$	0	1	2	3
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1				
2				1
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So, in the cell in row m *and column* n *we write the result of* $m +_4 n = m + n \ (mod\ 4)$.
For example the cell in row 2 *and column* 3 *is filled with* $2 + 3 \ (mod\ 4) = 1$.

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2	2	3	0	1
3	3	0	1	2

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- The inverse element to the element a is the one corresponding to the row and column where the neutral element e is placed.
- ...

Cayley table and latin square (1/4)

Question: Is it possible to recognize whether a table is a Cayley table of a group?
Answer: Almost.

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Theorem

The Cayley table of each group forms a latin square.

A latin square for a set M of n elements is a matrix $n \times n$ such that each row and column contains all elements of the set M.

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Unfortunately, not each Cayley table forming a latin square is a Cayley table of a group. Later we present a counterexample.

Cayley table and latin square (2/4)

Theorem

In each group, we can divide uniquely.
In other words: in each group $(G, \circ)$, for arbitrary $a, b \in G$ the equations

$$a \circ x = b \quad \textit{and} \quad y \circ a = b$$

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Since we are in a group, each element has only one inverse.
The only solutions of the equations are $x = a^{-1} \circ b$ and $y = b \circ a^{-1}$. □

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It is possible to prove that a group is a semigroup with a "unique division", i.e., the unique division guarantees the existence of a neutral element and inverse.

Cayley table and latin square (3/4)

Now we prove the theorem saying that the Cayley table of group is a latin square.

Proof.

Proof by contradiction.
Let us suppose that the table of some group $(G, \circ)$ is not a latin square.
Hence, in some row or column there is one element, denote it as b, repeated twice. WLOG[a], assume that it happens in row n and columns m_1 and m_2.

$\circ$	$\cdots$	m_1	$\cdots$	m_2	$\cdots$
$\vdots$		$\vdots$		$\vdots$	
n	$\cdots$	b	$\cdots$	b	$\cdots$
$\vdots$		$\vdots$		$\vdots$	

It follows that the equation $n \circ x = b$ has two different solutions, namely m_1 and m_2, which is a **contradiction with the previous theorem**! $\square$

[a] Without Loss Of Generality

Cayley table and latin square (4/4)

We have shown that the fact that a Cayley table is a latin square is a *necessary* condition for the given set and operation to be a group.

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The following example says it is not a *sufficient* condition.

Example

Let us consider a set $M = \{a, b, c\}$ *with operation given by the Cayley table:*

$\circ$	a	b	c
a	b	a	c
b	c	b	a
c	a	c	b

This table creates a latin square; in spite of it, it is not the table of a group (Why?!).

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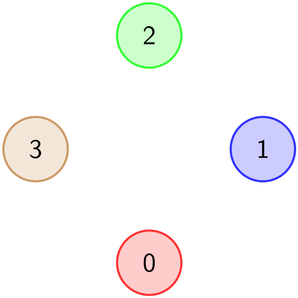

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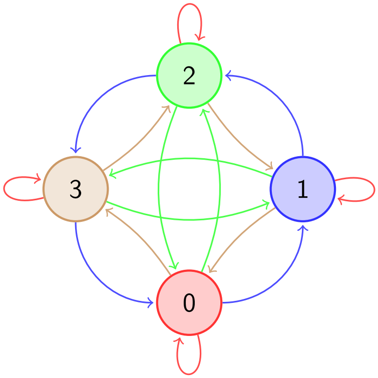

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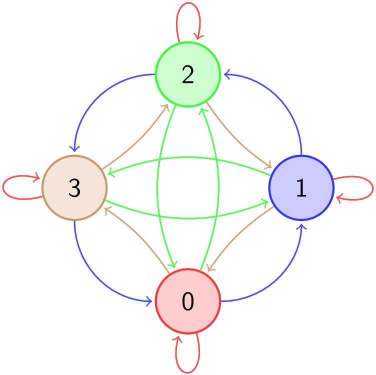

If the group in question is not Abelian, we need to depict edges (a, b) for $b = c \circ a$ for some $c \in M$.