

Mathematics for Informatics

Algebra: Homomorphisms, permutations
(lecture 6 of 12)

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B231 - Winter 2023/2024

created: November 1, 2023, 13:41

The same groups and distinct elements (1/5)

\mathbb{Z}_5^\times	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

\mathbb{Z}_4^+	0	1	2	3
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Aren't these two groups in fact the same group differing only in the "names" of their elements?

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- Now, it remains to rename only 2 and 3; we can check that both remaining possibilities work; we choose, for instance, $3 \leftrightarrow 1$ and $2 \leftrightarrow 3$.
- It suffices to reorder the rows... and we have the Cayley table of \mathbb{Z}_4^+ .

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This renaming is actually an **injective** mapping of the set $\{1, 2, 3, 4\}$ **onto** the set $\{0, 1, 2, 3\}$; let us denote it φ_1 :

$$\varphi_1(1) = 0, \quad \varphi_1(2) = 3, \quad \varphi_1(3) = 1, \quad \varphi_1(4) = 2.$$

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Would all bijections do the same job? And if not, what makes these two so special?

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Let us rename the elements of the group \mathbb{Z}_5^\times according to the bijection φ_3 :

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The bijection φ_3 does not give rise to the same structure of the group \mathbb{Z}_4^+ ; only φ_1 and φ_2 have this property.

The same groups and distinct elements (5/5)

The desired property, which only the bijections φ_1 and φ_2 have, is the following:

$$\text{for all } n, m \in \{1, 2, 3, 4\}, \text{ we have } \varphi(n \times_5 m) = \varphi(n) +_4 \varphi(m),$$

where \times_5 denotes the operation in the group \mathbb{Z}_5^\times , and $+_4$ the one in the group \mathbb{Z}_4^+ .

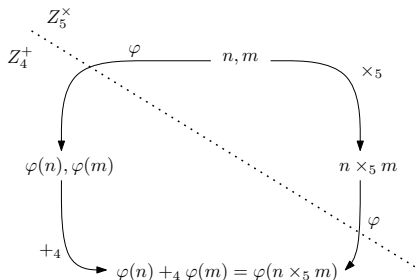
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*In words: If we apply the operation \times_5 to two arbitrary elements of the group \mathbb{Z}_5^\times and then we send the result to \mathbb{Z}_4^+ by φ , we obtain the same result as when we first transform by φ the elements to \mathbb{Z}_4^+ and **then** apply the operation $+_4$.*



Homomorphism and isomorphism

Definition

Let $G = (M, \circ_G)$ and $H = (N, \circ_H)$ be two groupoids. The mapping $\varphi : M \rightarrow N$ is a *homomorphism* from G to H if

$$\text{for all } x, y \in M, \text{ we have } \varphi(x \circ_G y) = \varphi(x) \circ_H \varphi(y).$$

If, moreover, φ is injective (resp. surjective, resp. bijective) we say that φ is a *monomorphism* (resp. *epimorphism*, resp. *isomorphism*).

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A homomorphism preserves the structure given by the binary operation: the result is the same if we first apply the operation and then the homomorphism or if we proceed inversely.

The only thing needed to define a homomorphism is that the set is closed under the binary operation; this is why we have defined homomorphism for the most general structures, i.e., groupoids.

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Isomorphic groups have the same order.

Fundamental properties of homomorphisms (1/2)

Theorem

Let φ be a homomorphism from a group $G = (M, \circ_G)$ to a group $H = (N, \circ_H)$.
The group $\varphi(G) = (\varphi(M), \circ_H)$ is a subgroup of H .

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- Denote by e_G the neutral element in G . Then $\varphi(e_G)$ is the neutral element in $\varphi(G)$ because, for all $x \in M$, we have $\varphi(e_G) \circ_H \varphi(x) = \varphi(e_G \circ_G x) = \varphi(x)$.

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- It can be shown similarly that the inverse of $\varphi(x)$ is $\varphi(x^{-1})$. □

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Example

$$\begin{aligned}\varphi : \mathbb{Z}_4^+ &\rightarrow \mathbb{Z}_8^+ \\ n &\mapsto 2n\end{aligned}$$

is a homomorphism and $\varphi(\mathbb{Z}_4^+)$ is the subgroup $\{0, 2, 4, 6\} \leq \mathbb{Z}_8^+$.

... up to isomorphism (1/2)

Isomorphic groups are in fact identical, they differ only in the names of their elements (as we have seen in the case of groups \mathbb{Z}_4^+ and \mathbb{Z}_5^\times).

If we say that there exists one group with a certain property **up to isomorphism**, it means that all groups with this property are isomorphic to each other.

We prove three well-known statements of this kind.

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Let $G = \langle a \rangle$ be a cyclic group with generator a .

We show that an arbitrary infinite cyclic group is isomorphic to the group $(\mathbb{Z}, +)$, and that an arbitrary cyclic group of order n is isomorphic to \mathbb{Z}_n^+ .

The rest follows from the transitivity of the relation “to be isomorphic”. □

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$(\mathbb{Z}, +)$ and \mathbb{Z}_n^+ are the only cyclic groups up to isomorphism.

... up to isomorphism (2/2)

The **Klein group** is the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \circ)$, where

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

and \circ is the component-wise addition modulo 2: e.g., $(1, 0) \circ (1, 1) = (0, 1)$.

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The Klein group is not cyclic and thus cannot be isomorphic to \mathbb{Z}_4^+ !

It is possible to show this (try it, it is easy):

Theorem

There exists only two groups of order 4 which are not isomorphic.

\mathbb{Z}_4^+ and the Klein group are the only two groups of order 4 up to isomorphism.

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- Each permutation $\pi \in \mathcal{S}_n$ can be defined by listing its values:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}.$$

The first row could be deleted, and so, e.g., $(1 \ 2 \ 4 \ 3 \ 5) \in \mathcal{S}_5$ is the permutation swapping elements **3** and **4**.

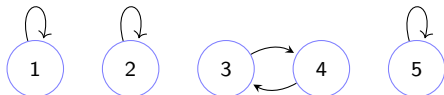
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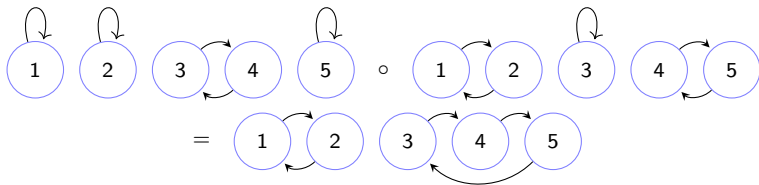
$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}.$$

The first row could be deleted, and so, e.g., $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ is the permutation swapping elements **3** and **4**.



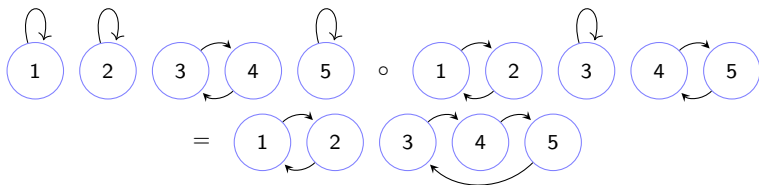
Permutations (2/2)

- Composition of permutations: $(1\ 2\ 4\ 3\ 5) \circ (2\ 1\ 3\ 5\ 4) = (2\ 1\ 4\ 5\ 3)$.



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- The composition of permutations is associative, the permutation $(1\ 2\ 3\ \cdots\ n)$ is the neutral element, and the inverse element is the inverse permutation. Hence, \mathcal{S}_n is a group of order $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

Group of permutations

Subgroups of the symmetric group \mathcal{S}_n are called **groups of permutations**.

Example

The permutation $(1\ 2\ 4\ 3\ 5) \in \mathcal{S}_5$ swapping the elements 3 and 4 generates a subgroup of \mathcal{S}_5 containing two elements: $(1\ 2\ 4\ 3\ 5)$ and $(1\ 2\ 3\ 4\ 5)$.

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The structure of the subgroups of \mathcal{S}_n is very (in some sense maximally) rich:

Theorem (Cayley)

Each finite group is isomorphic to some group of permutations.

Proof: hint only for interested.

Let a be an element of a group G of order n with a binary operation \circ . Put $\pi_a(x) = a \circ x$. Since in any group we can divide uniquely, π_a is a bijection and thus a permutation. The desired monomorphism is the mapping defined for each element a in this way: $\varphi(a) = \pi_a$. □