

# NIE-MPI: Tutorial 5

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## 5.1 Subgroups

**Exercise 5.1.** Which of the following sets forms a subgroup of the group  $(\mathbb{Q} \setminus \{0\}, \cdot)$ ?

- (a) The set of all even numbers without zero;
- (b) The set of all odd numbers;
- (c)  $\{2^n : n \in \mathbb{Z}\}$ ;
- (d)  $\{2^n \cdot 3^m : n, m \in \mathbb{Z}\}$ ;
- (e)  $\left\{\frac{1+2n}{1+2m} : n, m \in \mathbb{Z}\right\}$ .

**Exercise 5.2.** Find some other subgroup(s) of the group  $(\mathbb{Q} \setminus \{0\}, \cdot)$  distinct from the ones in the previous exercise.

**Exercise 5.3.** Find all subgroups of the group given by following Cayley table:

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

**Hint:** when looking for a subgroup, be more specific on what subgroup you look for.

**Exercise 5.4.** Is  $G = (M, \cdot)$  with  $M = \{a + b\sqrt{2} : a, b \in \mathbb{Q}, a \neq 0 \vee b \neq 0\}$  a subgroup of the group  $(\mathbb{R} \setminus \{0\}, \cdot)$ ?

**Exercise 5.5.** Specify the following subgroups of  $(\mathbb{Z}, +)$ .

- (a)  $\langle 2 \rangle$ ;
- (b)  $\langle 5 \rangle$ ;
- (c)  $\langle \{2, 3\} \rangle$ ;
- (d)  $\langle \{2, 4\} \rangle$ ;
- (e)  $\langle \{6, 12\} \rangle$ ;
- (f)  $\langle \{n, m, \ell\} \rangle$  for  $n, m, \ell \in \mathbb{N}^+$ .

## 5.2 Cyclic groups and generators

**Exercise 5.6.** Find all generators and all subgroups of  $\mathbb{Z}_{11}^\times = (\mathbb{Z}_{11} \setminus \{0\}, \cdot_{11})$ .

**Exercise 5.7.** Find all generators and all subgroups of  $\mathbb{Z}_{13}^\times = (\mathbb{Z}_{13} \setminus \{0\}, \cdot_{13})$ . Find the inverse elements of 12, 5, and 11.

**Exercise 5.8.** What is the probability that an element of  $\mathbb{Z}_{23}^\times$  chosen randomly is a generator?

**Exercise 5.9.** Is the number 5 a generator of the group  $\mathbb{Z}_{23}^\times$ ?

What is the least number of “computation steps” needed to decide this question?