BIE-PST	**Written Final Exam**	Winter 2023/24
Probability and Statistics	FIT CTU in Prague	Jan 09, 2024

Given name: ______________________ **Family name:** ______________________

User ID: ______________________ **Signature:** ______________________

Instructions

You have 90 minutes to complete the exam. You are **not allowed to cooperate or consult with anyone** while working on the exam. Remember that copying is easy to detect and that CTU in Prague considers cheating a very serious issue.

You must **show all your work and justify your methods** to obtain full score. Simplify your answers. You do not need to evaluate expressions such as e^3, $\ln 5$, $\sqrt{3}$, 2π. Do not use scratch paper. Use the back of the previous page if additional space is needed. **Calculators, organizers or computers are not allowed.** During the exam all cell phones and pagers brought into the classroom must be in your backpack, put in "silent" mode. The exam is worth a total of 60 points. A minimum of 30 points is needed to pass.

Achieved points:

Question	Max	Yours
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:		

1. There are 5 people at a train platform. A train with 3 wagons comes and each of the people chooses one wagon to get in at random, independently on other people.

   (a) What is the probability that there will be exactly two people in the first wagon?

   (b) What is the probability that there will be at least four people in the first wagon?

2. Let X be the result of the following process: We roll a balanced six-sided die and if we roll a number less than six, then X is equal to this number. If we roll six, we roll once again and X is the result of the second roll.

(a) Find the distribution of the random variable X.

(b) Find the expected value of X, i.e. $\mathrm{E}\,X$.

(c) What is the probability that in 10 rolls we obtain exactly three twos? (It is sufficient to write the correct formula.)

3. Consider a discrete random vector (X, Y). The *conditional distribution* of X for given $Y = y$ is given as:

$\mathrm{P}(X = x \mid Y = y)$		X			
		0	1	2	3
Y	0	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{5}{11}$	$\frac{1}{11}$
	1	$\frac{9}{20}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$

Further we know that $\mathrm{P}(Y = 0) = \frac{11}{21}$.

(a) Find the joint distribution of the random vector (X, Y).

(b) Are X and Y independent? Explain thoroughly.

(c) Find the conditional expectation $\mathrm{E}(X \mid Y = 0)$.

(d) Find the expectation $\mathrm{E}(X)$.

4. We are operating a FTP server with 10 000 users. Suppose that the expected volume of stored data of each user is 20 GB, with a standard deviation of 8 GB.

   (a) Find the expectation and variance of the total volume of data stored on the server.

   (b) How large should be the hard drive of the server, so that it will be sufficient for all stored data with a probability of 99 %? *Use the Central Limit Theorem!*

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution with unknown parameters.

   (a) Explain the maximum likelihood method of parameter estimations, step by step.

   (b) Using the maximum likelihood method find the point estimate of the parameter p of the geometric distribution.
   Hint: For the geometric distribution it holds that
   $\mathrm{P}(X = k) = (1 - p)^{k-1}p,\ k \in \mathbb{N}.$

6. According to the menu of an unnamed restaurant, the expected weight of steaks should be 230 grams. We took a sample of hundred independent steaks, measured their weights and obtained:

$$\sum_{i=1}^{n} X_i = 20000, \qquad \sum_{i=1}^{n} X_i^2 = 4990000.$$

Suppose that the weight of steaks is normally distributed.

(a) Find point estimates of the expectation and variance of the weight of the steaks.

(b) Design a statistical test which can be used to decide whether the expected weight of steaks is truly 230 grams, or whether it is significantly smaller. Perform the test on level of significance 5 $\%$. Explain your steps in detail. In computations, use a suitable approximation or rounding.