BIE-PST – Probability and Statistics

Lecture 6: Random vectors I.

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6 Random vectors

Sometimes we can measure *several* random *variables* at once from *one result* of an experiment.

The individual variables can have different distributions and the values of the variables can be strongly mutually interconnected. It is appropriate to *describe* their *distribution together* as the so called *joint distribution.*

Definition 6.1. Consider two random variables X and Y defined on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We define their *joint distribution function* $F_{X,Y}(x, y)$ as

$$F_{X,Y}(x, y) = \mathrm{P}(X \leq x \cap Y \leq y).$$

For n random variables $X_1, X_2, \ldots, X_n \overset{\text{denote}}{=} \boldsymbol{X}$ we define the joint distribution function as

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \mathrm{P}(X_1 \leq x_1 \cap \ldots \cap X_n \leq x_n).$$

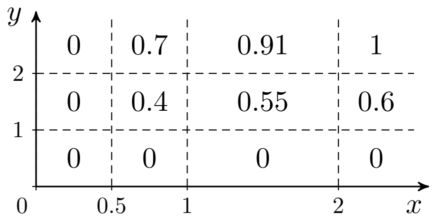
We use the notation $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{X} = (X_1, \ldots, X_n)$.

The couple (X, Y) or, n-tuple $(X_1, \ldots, X_n)$, is called a *random vector.*

Example 6.2. Let X and Y be random variables with a joint discrete distribution given by the following probabilities:

			x	
$\mathrm{P}(X = x \cap Y = y)$		0.5	1	2
y	1	0.4	0.15	0.05
	2	0.3	0.06	0.04

Compute the joint distribution function $F_{X,Y}(x, y) = \mathrm{P}(X \leq x \cap Y \leq y)$*:*



The joint distribution function has analogous properties as the distribution function of one variable.

Theorem 6.3. *The joint distribution function $F_{X,Y}$ of random variables X and Y has following properties:*

i) if $x_1 < x_2$ and $y_1 < y_2$ then $F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$.

ii) $\forall y \in \mathbb{R}, \ \lim_{x \to -\infty} F_{X,Y}(x, y) = 0$ *and* $\forall x \in \mathbb{R}, \ \lim_{y \to -\infty} F_{X,Y}(x, y) = 0.$

iii) $\forall y \in \mathbb{R}, \ \lim_{x \to +\infty} F_{X,Y}(x, y) = F_Y(y)$ *and* $\forall x \in \mathbb{R}, \ \lim_{y \to +\infty} F_{X,Y}(x, y) = F_X(x).$

Proof. Analogously as for the distribution function of one random variable. □

6.1 Vectors of discrete random variables

A *distribution* of random *variables* X and Y on the same probability space is described by the *joint distribution function*

$$F_{X,Y}(x,y) = \mathrm{P}(X \le x \cap Y \le y).$$

If the variables X and Y are *discrete*, it is often useful to describe the distribution by the *joint probabilities* of their values.

Definition 6.4. *The joint probabilities of values* of two discrete random variables X and Y is

$$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(\{X = x\} \cap \{Y = y\}).$$

Taken as a function of x and y, the probabilities are called the *joint probability mass function.*

The *joint distribution function* of two discrete random variables X and Y is

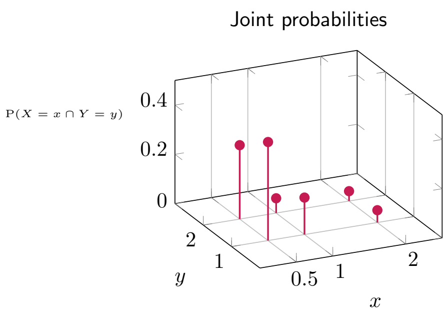
$$F_{X,Y}(x,y) = \mathrm{P}(X \le x \cap Y \le y) = \sum_{\{i:\, x_i \le x\}} \sum_{\{j:\, y_j \le y\}} \mathrm{P}(X = x_i \cap Y = y_j)$$

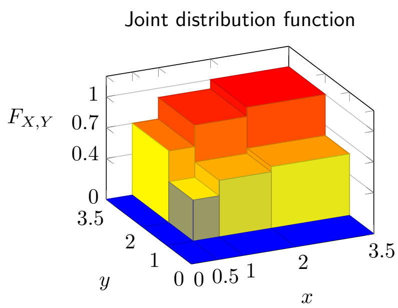
From this it follows that $F_{X,Y}(x,y)$ has a stepwise structure.

The *normalization condition* follows from the properties of the joint distribution function:

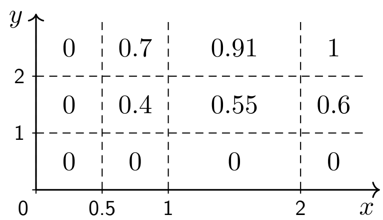
$$\begin{aligned}
&\sum_i \sum_j \mathrm{P}(X = x_i \cap Y = y_j) = \sum_i \mathrm{P}\left(\{X = x_i\} \cap \textstyle\bigcup_j \{Y = y_j\}\right) \\
&= \sum_i \mathrm{P}\left(\{X = x_i\} \cap \{Y \in \mathbb{R}\}\right) = \sum_i \mathrm{P}\left(X = x_i\right) \\
&= \mathrm{P}\left(\textstyle\bigcup_j \{X = x_i\}\right) = \mathrm{P}\left(\{X \in \mathbb{R}\}\right) = \mathrm{P}(\Omega) = 1.
\end{aligned}$$

Example 6.5.





$\mathrm{P}(X = x \cap Y = y)$		x: 0.5	1	2
y	2	0.3	0.06	0.04
	1	0.4	0.15	0.05



Marginal distribution

Sometimes we have the joint distribution of variables X and Y, but we are *not interested in* the *values of* Y. From the joint distribution function $F_{X,Y}$ we would then want to obtain only the distribution function F_X of the variable X.

The distribution obtained this way is called the *marginal* distribution of random variable X.

Theorem 6.6. *Let* $\mathrm{P}(X = x \cap Y = y)$ *be the joint probabilities of values of two discrete variables* X *and* Y*. The* marginal distribution *(or* marginal probabilities*) of a* X *is given by*

$$\mathrm{P}(X = x) = \sum_j \mathrm{P}(X = x \cap Y = y_j).$$

Proof. The events $\{Y = y_j\}$ for $j = 1, 2, \dots$ create a countable partition of Ω. From this follows:

$$\mathrm{P}(X = x) = \mathrm{P}(\{X = x\} \cap \{Y \in \mathbb{R}\}) = \mathrm{P}(\{X = x\} \cap (\textstyle\bigcup_j \{Y = y_j\})) =$$
$$= \mathrm{P}\,\Big(\bigcup_j (\{X = x\} \cap \{Y = y_j\})\Big) = \sum_j \mathrm{P}(\{X = x\} \cap \{Y = y_j\}).$$

□

Example 6.7. Let X and Y be two random variables with the following joint distribution:

$\mathrm{P}(X = x \cap Y = y)$		x = 0.5	x = 1	x = 2	$\mathrm{P}(Y = y)$
y	2	0.3	0.06	0.04	0.4
	1	0.4	0.15	0.05	0.6
$\mathrm{P}(X = x)$		0.7	0.21	0.09	

Find the marginal distribution of X *and* Y *separately (find the marginal probabilities* $\mathrm{P}(X = x)$ *and* $\mathrm{P}(Y = y)$*.)*

$$\mathrm{P}(Y = y) = \begin{cases} 0.6 & \text{for } y = 1 \\ 0.4 & \text{for } y = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mathrm{P}(X = x) = \begin{cases} 0.7 & \text{for } x = 0.5 \\ 0.21 & \text{for } x = 1 \\ 0.09 & \text{for } x = 2 \\ 0 & \text{elsewhere} \end{cases}$$

6.2 Independence of discrete random variables

Similarly as with random events, we want to be able to determine, whether the knowledge of one variable changes in some way the distribution of an other one.

Definition 6.8. Random variables X and Y are called independent if for all $x,\ y \in \mathbb{R}$ the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent. Equivalently, if it holds that for all $x, y \in \mathbb{R}$

$$\mathrm{P}(X \leq x \cap Y \leq y) = \mathrm{P}(X \leq x) \cdot \mathrm{P}(Y \leq y).$$

Random variables $X_1, \ldots, X_n$ are called independent if for all $\boldsymbol{x} \in \mathbb{R}^n$ it holds that

$$\mathrm{P}(\boldsymbol{X} \leq \boldsymbol{x}) = \prod_{i=1}^{n} \mathrm{P}(X_i \leq x_i).$$

Random variables forming a countable collection $X_1, X_2, \ldots$ are called independent if all finite n-tuples $X_{i_1}, \ldots, X_{i_n}$ are independent.

For discrete random variables we can verify the independence by means of the probabilities of values:

Theorem 6.9. *Discrete random variables X and Y are independent if for all $x,\ y \in \mathbb{R}$ the events $\{X = x\}$ and $\{Y = y\}$ are independent. Equivalently, if it holds that for all $x, y \in \mathbb{R}$*

$$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(X = x) \cdot \mathrm{P}(Y = y).$$

Random variables $X_1, \ldots, X_n$ are independent if for all $\boldsymbol{x} \in \mathbb{R}^n$ it holds that

$$\mathrm{P}(\boldsymbol{X} = \boldsymbol{x}) = \prod_{i=1}^{n} \mathrm{P}(X_i = x_i).$$

Proof. If the condition regarding equalities holds, it must hold also for all inequalities, because they can be rewritten as sums of probabilities of disjoint events.

If the condition regarding inequalities hold, it must hold also for all equalities, because the difference of probabilities of inequalities yields probabilities of equalities. □

Example 6.10 (– continuation)**.** Random variables X and Y have the following joint and marginal distributions:

		x			
	$\mathrm{P}(X = x \cap Y = y)$	0.5	1	2	$\mathrm{P}(Y = y)$
y	2	0.3	0.06	0.04	0.4
	1	0.4	0.15	0.05	0.6
	$\mathrm{P}(X = x)$	0.7	0.21	0.09	

Are X and Y independent?

No, they are not independent because e.g. for $x = 0.5$ and $y = 2$ it holds that

$$0.3 = \mathrm{P}(X = 0.5 \cap Y = 2) \neq \mathrm{P}(X = 0.5) \cdot \mathrm{P}(Y = 2) = 0.7 \cdot 0.4 = 0.28.$$

6.3 Vectors of continuous random variables

The *distribution* of random *variables* X and Y on the same probability space is described by the *joint distribution function*

$$F_{X,Y}(x, y) = \mathrm{P}(X \leq x \cap Y \leq y).$$

If the variables X and Y are *continuous*, it is often useful to describe the distribution by the *joint probability density.*

Definition 6.11. Two random variables X and Y have a *joint (absolutely) continuous* distribution if there exists a **non-negative** function $f_{X,Y} : \mathbb{R}^2 \to [0, +\infty)$ such that for all $x, y \in \mathbb{R}$ it holds

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, \mathrm{d}u\, \mathrm{d}v.$$

The function $f_{X,Y}$ is called the *joint probability density* of the random variables X, Y or of the random vector (X,Y).

Similarly as in the one-dimensional case it holds that:

- Where the derivative exists:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y).$$

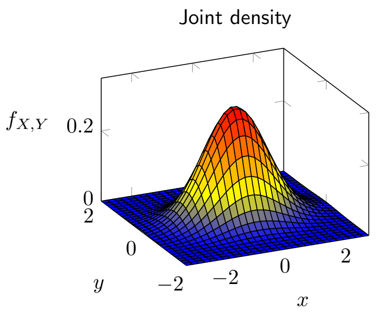
- The *joint distribution function* is *continuous.*
- *Normalization condition*: $\displaystyle\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\ \mathrm{d}x\ \ \mathrm{d}y = 1$
- For all $x, y \in \mathbb{R}$ and all Borel sets A, B on $\mathbb{R}$

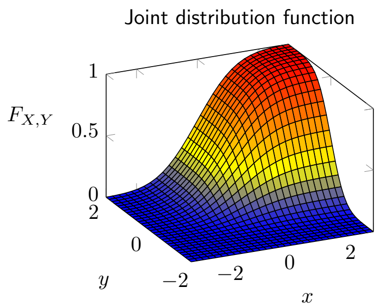
$$\mathrm{P}(X = x \cap Y \in B) = \mathrm{P}(X \in A \cap Y = y) = \mathrm{P}(X = x \cap Y = y) = 0.$$

- $\displaystyle\mathrm{P}(\{a < X \le b\} \cap \{c < Y \le d\}) = \int_{c}^{d} \int_{a}^{b} f_{X,Y}(x,y)\ \mathrm{d}x\ \ \mathrm{d}y.$
- For all B Borel subset of $\mathbb{R}^2$ (meaning that $\{X \in B\}$ is an event)

$$\mathrm{P}\left((X,Y) \in B\right) = \iint_B f_{X,Y}(x,y)\ \mathrm{d}x\ \ \mathrm{d}y.$$

Example 6.12.





$$f_{X,Y}(x,y) = \frac{1}{\pi} e^{-\frac{x^2}{2} - 2y^2} \qquad F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} \frac{1}{\pi} e^{-\frac{u^2}{2} - 2v^2}\ \mathrm{d}u\, \mathrm{d}v$$

Marginal distribution

For computing the *marginal* distribution of two variables X and Y from the joint density we can use a formula analogous to the discrete case:

Theorem 6.13. *Let X and Y be two random variables having a joint continuous distribution with joint density $f_{X,Y}$. Then X and Y are both continuous too, and the* marginal densities *f_X, f_Y are given by*

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}y, \qquad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}x.$$

Proof. We know that:

$$F_X(x) = \mathrm{P}(X \le x) = \mathrm{P}(X \le x \cap Y \in \mathbb{R}) = \int_{-\infty}^{x} \left(\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\ \mathrm{d}v \right) \mathrm{d}u\,.$$

The statement of the theorem is obtained by differentiating with respect to x, or by comparing this formula to the definition of the distribution function of a continuous random variable. The second part is analogous. □

6.4 Independence of continuous random variables

The independence of continuous random variables can be determined by means of densities.

Theorem 6.14. *Two continuous random variables X and Y are called independent if and only if for all x, $y \in \mathbb{R}$ the following equality holds*

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y).$$

Random variables $X_1, \ldots, X_n$ are called independent if for all $\boldsymbol{x} \in \mathbb{R}^n$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{n} f_{X_i}(x_i).$$

Proof. Two random variables X and Y are independent if

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y).$$

Taking the derivatives of both sides with respect to both x and y yields one implication. Integrating both sides of the equality for densities yields the other direction. □

Remark 6.15. While verifying the independence of X and Y we can use the following: Consequence: If it is possible to *decompose* $f_{X,Y}$ to

$$f_{X,Y}(x,y) = g(x) \cdot h(y), \quad \forall x, y \in \mathbb{R},$$

where $g(x)$ and $h(y)$ are non-negative functions, then the variables X and Y are *independent.* Note that the function $g(x)$ does not need to be the density $f_X(x)$, and the function $h(y)$ to be the density $f_Y(y)$; they can differ from densities by a multiplicative constant. Thus in case we find $f_{X,Y}(x,y) = g(x) \cdot h(y)$, we know that the random variables X and Y are independent and that their densities are respectively $f_X(x) = k \cdot g(x)$ and $f_Y(y) = \frac{1}{k} h(y)$, where the constant k must be computed from the normalization condition.

✓ *Do the proof yourself by inserting into the formula for marginal densities.*

The statement of the consequence can be formulated for independence of a general random vector $X_1, \ldots, X_n$ too.

Example 6.16. Let X and Y random variables having the joint probability density

$$f_{X,Y}(x,y) = ye^{-2x} \qquad \text{for } x \in [0,+\infty) \text{ and } y \in [0,2].$$

Are the variables X and Y independent?
Marginal densities:

$$f_X(x) = \int_0^2 ye^{-2x}\,\mathrm{d}y = e^{-2x}\int_0^2 y\,\mathrm{d}y = e^{-2x}\left[\frac{y^2}{2}\right]_0^2 = e^{-2x}\left(\frac{4}{2}-0\right) = 2e^{-2x}.$$

$$f_Y(y) = \int_0^{+\infty} ye^{-2x}\,\mathrm{d}x = y\int_0^{+\infty} e^{-2x}\,\mathrm{d}x = y\left[\frac{e^{-2x}}{-2}\right]_0^{+\infty} = y\left(0-\frac{1}{-2}\right) = \frac{y}{2}.$$

Independence:

$$ye^{-2x} = f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = 2e^{-2x}\cdot\frac{y}{2} = ye^{-2x}.$$

Yes, they are independent!

6.5 Conditional distribution

6.5.1 Discrete conditional distribution

Now we will study the *distribution* of a random variable X under the assumption that we *know* the value of the variable $Y = y$.

Suppose that we have a partial information about the result of an experiment and we are interested in the change in our prediction.

It is reasonable to introduce the conditional distribution by means of the *conditional probability* under the condition of the event $\{Y = y\}$.

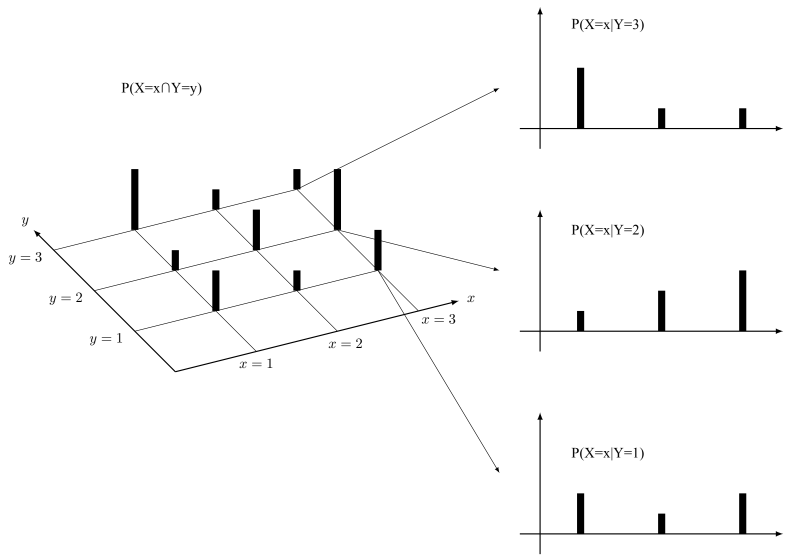
Definition 6.17. Let $\mathrm{P}(Y = y) > 0$. Then, the *conditional distribution function* $F_{X|Y}(\cdot|y)$ of the variable X given $Y = y$ is defined as

$$F_{X|Y}(x|y) = \mathrm{P}(X \le x | Y = y).$$

The *conditional probabilities of values* of X given (under the condition of) $Y = y$ are given, analogously, by

$$\mathrm{P}(X = x | Y = y).$$

Illustration of conditional probabilities $\mathrm{P}(X = x|Y = y)$



From the definition it follows that:

$$\mathrm{P}(X = x|Y = y) = \frac{\mathrm{P}(X = x \cap Y = y)}{\mathrm{P}(Y = y)}.$$

Definition 6.18. Let $\mathrm{P}(Y = y) > 0$. The expectation of the variable X with conditional probabilities $\mathrm{P}(X = x|Y = y)$ is called the *conditional expectation* of X given $Y = y$ and is denoted as $\mathrm{E}(X|Y = y)$.

Thus it holds that:

$$\mathrm{E}(X|Y = y) = \sum_i x_i \,\mathrm{P}(X = x_i|Y = y) = \sum_i x_i \frac{\mathrm{P}(X = x_i \cap Y = y)}{\mathrm{P}(Y = y)}.$$

6.5.2 Continuous conditional distribution

When observing two *continuous* random variables X and Y, *it is not possible to use an event* $\{Y = y\}$ *as a condition*, because $\mathrm{P}(Y = y) = 0$.

The conditional distribution can be obtained using a limit approach: Let $f_{X,Y}$ be joint density of X, Y and it holds $f_Y(y) > 0$. Then for $\Delta y << 1$

$$\begin{aligned}\mathrm{P}(X \le x \mid y \le Y \le y + \Delta y) &= \frac{\mathrm{P}(X \le x \cap y \le Y \le y + \Delta y)}{\mathrm{P}(y \le Y \le y + \Delta y)} = \\ &= \frac{\int_{-\infty}^{x} \int_{y}^{y+\Delta y} f_{X,Y}(u, v)\,\mathrm{d}v\,\mathrm{d}u}{\int_{y}^{y+\Delta y} f_Y(v)\,\mathrm{d}v} \approx \frac{\int_{-\infty}^{x} f_{X,Y}(u, y)\Delta y\,\mathrm{d}u}{f_Y(y)\Delta y} = \\ &= \int_{-\infty}^{x} \frac{f_{X,Y}(u, y)}{f_Y(y)}\,\mathrm{d}u.\end{aligned}$$

After taking a limit $\Delta y \to 0$ we intuitively obtain the result as

$$\mathrm{P}(X \le x | Y = y) = \int_{-\infty}^{x} \frac{f_{X,Y}(u,y)}{f_Y(y)} \,\mathrm{d}u.$$

the previous inference lead us to the following formal definition:

Definition 6.19. The *conditional distribution function* of a variable X given (under the condition of) $Y = y$ is defined as

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} \frac{f_{X,Y}(u,y)}{f_Y(y)} \mathrm{d}u,$$

for all y such that $f_Y(y) > 0$. We use the notation $\mathrm{P}(X \le x | Y = y) = F_{X|Y}(x|y)$, too.

The conditional density is defined accordingly:

Definition 6.20. The *conditional probability density* of X given (under the condition of) $Y = y$ is given as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

for all y such that $f_Y(y) > 0$.

Analogously as in the discrete case we define the *conditional expectation* for continuous random variables:

Definition 6.21. Let $f_Y(y) > 0$. The expectation of variable X with density $f_{X|Y}(x|y)$ is called the *conditional expectation* of X given $Y = y$ and is denoted as $\mathrm{E}(X|Y = y)$.

We compute the conditional expectation for a given value y as follows:

$$\mathrm{E}(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) \,\mathrm{d}x = \int_{-\infty}^{\infty} x \frac{f_{X,Y}(x,y)}{f_Y(y)} \,\mathrm{d}x = g(y),$$

where g is a function which arises from the integration.

