BIE-PST – Probability and Statistics

Lecture 9: Parameter point estimators

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9 Basic notions of statistics

9.1 Statistical inference

So far we have dealt with *probabilistic* problems with known parameters. For example if we have a box with r red and b blue balls, we can:

- find the probability of drawing a blue ball,
- find the probability of drawing a certain number of blue balls in three draws with or without replacement,
- find the expected number of blue balls in 10 draws with replacement,
- make statements about a sequence of 1000 draws,
- etc.

Now we will deal with *statistical* problems. For example if we have a box with an unknown number of red and blue balls, we can take a sample and:

- estimate the proportion of red and blue balls,
- test whether there are 50% of blue balls or more,
- test whether the red/blue proportion is the same among two separate boxes,
- etc.

Probability theory deals with mathematical models of processes (experiments, tests, etc.) with random results. These *models* are then *utilized for prediction* of possible *outcomes*, i.e., we determine probabilities of events, distributions and expected values of random variables, etc.

Mathematical statistics proceeds, to some extent, reversely. *On the grounds* of real *outcomes we choose* an appropriate model and *estimate* its parameters. Then we can *test* hypotheses about these parameters and *verify* how well does the model fit the data.

9.1.1 Random sample

Statistics uses specific terminology.

Definition 9.1. An n-tuple of independent and identically distributed random variables (i.i.d.) $X_1, \ldots, X_n$ with distribution function F is called a *random sample* from the distribution F.

Examples 9.2.

- Measurement of a given variable in n independent repetitions of some experiment.
- Time to execute an algorithm in n repeated runs.
- Measurement of body height of n different people.

Definition 9.3. The *random sample realization* (random vector of observations or simply *data*) is an n-tuple of particular observed values $x_1, \ldots, x_n$.

9.1.2 Steps of statistical inference

Consider a random sample from an unknown distribution. On the grounds of measured data (random sample realizations) we want to learn as much as possible about the underlying distribution.

Typical steps of statistical inference:

- *Estimate the shape of the distribution* – restrict the inference to a family of distributions F_θ with a parameter θ. This can follow from prior knowledge, intuition or experience.
- *Estimate the parameters of the distribution*
  - *Point estimation* – determine the "most probable" value of θ.
  - *Interval estimation* – determine an interval (region) in which θ lies with a given large probability.
- *Verify the model – hypothesis testing*
  - *Goodness-of-fit tests* – we verify hypothesis about the shape of the probability distribution –e.g., whether the investigated variable has the normal distribution.
  - *Parametric tests* – we state a hypothesis about the parameter θ (e.g., $\theta = 0$) and on the grounds of measured data we try to decide whether this hypothesis can be true or not.

9.2 Estimation of the shape of the distribution

The *distribution* of an investigated random variable usually *may not be* absolutely *arbitrary.*

Based on previous experience, intuition or the type of underlying data we can often

- determine whether the variable is discrete or continuous;
- approximate the shape of the distribution (e.g., exponential, normal, etc.);
- establish other possible determining properties (e.g., range of values, zero expectation, etc.).

This information leads us to a *choice* of a particular *model*, thus to the

- choice of parametric distribution family $\{F_\theta(x) | \theta \in \Theta\}$, where Θ is a set of all possible values of parameter θ;
- and the *assumption* that our random sample is governed by distribution from this family.

Examples of possible models

- *Bernoulli distribution* – tossing with an unknown coin

$$\{\mathrm{Be}(p) \,|\, p \in [0, 1]\}$$

  Parameter $\theta = p$ and $\Theta = [0, 1]$.

- *Exponential distribution* – times between two incoming request on a database server

$$\{\mathrm{Exp}(\lambda) \mid \lambda \in (0, +\infty)\}$$

  Parameter $\theta = \lambda$ and $\Theta = (0, +\infty)$.

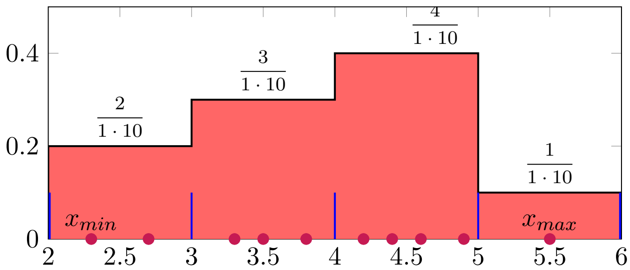
- *Normal distribution* – results of an IQ test in a given population

$$\{\mathrm{N}(\mu, \sigma^2) \mid \mu \in (-\infty, +\infty), \sigma^2 \in (0, +\infty)\}$$

  Two dimensional parameter $\theta = (\mu, \sigma^2)$ and $\Theta = (-\infty, +\infty) \times (0, +\infty)$.

9.2.1 Histogram

The shape of the *density* can be estimated by the *histogram*:

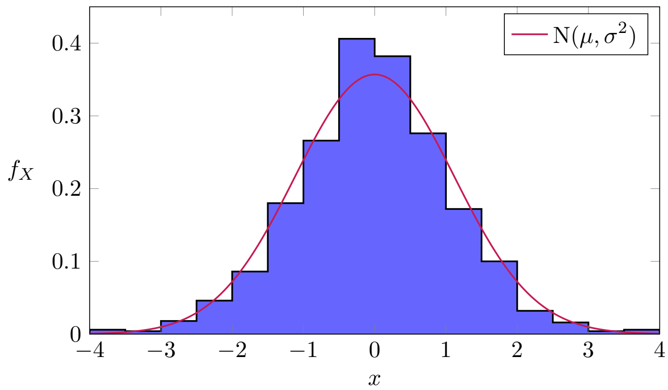


- Determine the data range.
- Choose a number of bins k and their size h (here $k = 4$ and $h = 1$).
- Over each bin, plot a column of the size

$$\frac{\text{number of observation in bin}}{h \cdot \text{total number of observations}} \overset{\text{denote}}{=} \frac{m_i}{h \cdot n}.$$

Notice that the area under a histogram is 1. It is thanks to dividing by total number of observations n. (It is not always done, we must beware while using a software.) In this case it is possible to visually compare the histogram with known densities. However, there remains the questions of how to choose the number and location of bins, their size, or decide whether the bins should be of equal size or not. This is often a difficult question and inappropriate choice can destroy the histogram. Very reliable visualisation (and estimate as well) is always the empirical distribution function.

Example 9.4. We measured 1000 values from an unknown distribution. The histogram of these values is:



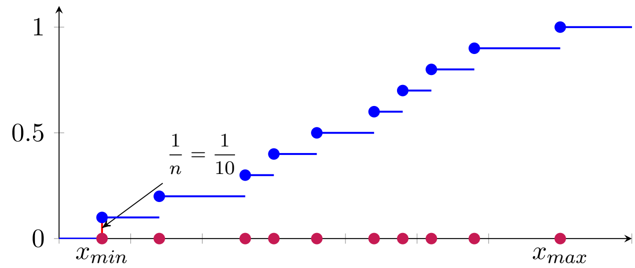
We can assume that we deal with values from the *normal distribution* with unknown parameters μ and σ^2.

9.2.2 Empirical distribution function

The shape of the *distribution function* can be estimated by the *empirical distribution function*:

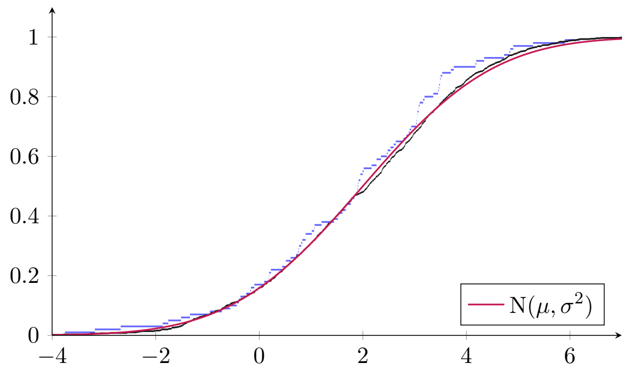
$$F_n(x, X_1, \ldots, X_n) = F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq x\}}.$$

In other words, the probability that the random variable in question is less than or equal x can be estimated by the proportion of data points which are less than or equal to x.



✓ *The empirical distribution function is a piecewise constant function with jumps of size* $\frac{1}{n}$ *in the observed data points.* Later we will see that the empirical distribution function is a good estimate of an unknown distribution function of the random sample.

Example 9.5. We measured 100 and 1000 values from an unknown distribution. The empirical distribution functions are:



We can assume that we deal with values from the *normal distribution* with unknown parameters μ and σ^2.

Example 9.6 (– waiting for a bus)**.** Every morning we measure the time which we spend waiting for a bus on our way to school. After 15 days, we have observed the following data (in minutes, sorted):

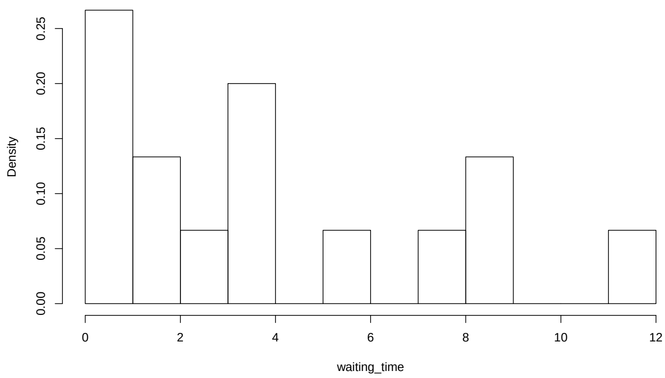
0.1 0.3 0.5 0.7 1.0 1.9 2.8 3.4 3.5 3.8 5.3 7.7 8.6 8.7 11.1

Suppose that the waiting times form a random sample $(X_1, \ldots, X_{15})$ from an unknown distribution *Find the histogram and the empirical distribution function of this distribution.*

The data are in the interval $[0, 12]$. If we take the bandwidth h too small or too large, the histogram may be inaccurate:

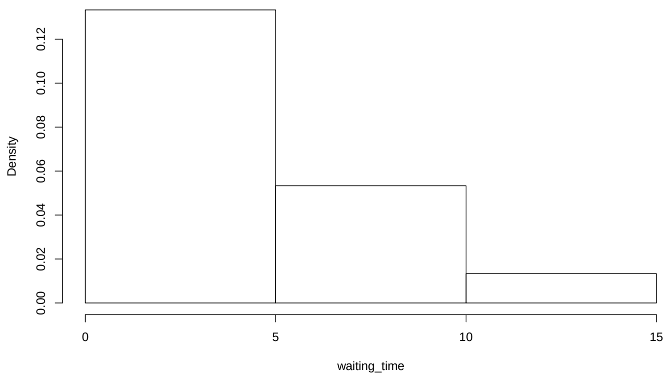
0.1 0.3 0.5 0.7 1.0 1.9 2.8 3.4 3.5 3.8 5.3 7.7 8.6 8.7 11.1

```
>hist(waiting_time,prob=T,breaks=12)
```



The bandwidth seems too small.

```
>hist(waiting_time,prob=T,breaks=2)
```

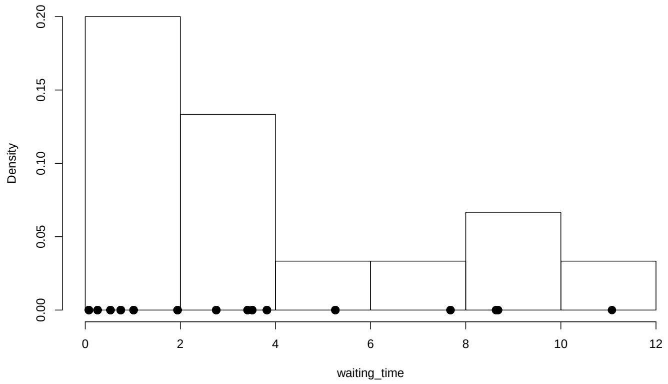


The bandwidth seems too large.

The data are in the interval $[0, 12]$. It seems reasonable to divide them into six parts, each covering two minutes. Each data point constitutes $\frac{1}{h \cdot n} = \frac{1}{2 \cdot 15} = 0.0\bar{3}\bar{3}$:

0.1 0.3 0.5 0.7 1.0 1.9 2.8 3.4 3.5 3.8 5.3 7.7 8.6 8.7 11.1

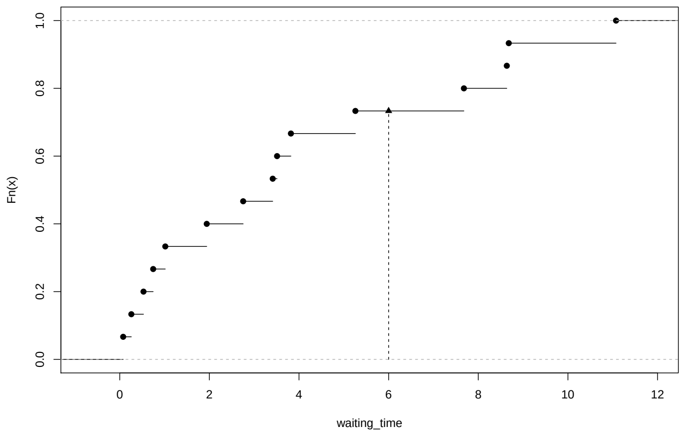
```
>hist(waiting_time,prob=T)
```



The histogram might seem similar to the exponential distribution.

We proceed from the left and add a jump of 1/15 at each data point encountered:

```
>plot(ecdf(waiting_time))
```



Now we can estimate probabilities of the type $\mathrm{P}(X \leq x)$ using $F_n(x)$. The probability that we do not need to wait for more than six minutes is estimated as $F_n(6) = 11/15 \doteq 0.733$, which is the proportion of data points less than or equal to 6.

The *quantiles* q_α divide the population so that there are α% of values under the α-quantile and $(1-\alpha)$% above. The 50%-quantile is called the *median* and divides the population into two equally large parts with respect to probability. If we denote the ordered data as

$$\left(x_{(1)}, x_{(2)}, \dots, x_{(n)}\right),$$

the α%-quantile can be estimated as $x_{(\lceil n\alpha \rceil)}$. This is then the inverse of the empirical distribution function. The median $q_{0.5}$ can then be estimated as the *middle value* of the ordered data, $x_{(\lceil \frac{n}{2} \rceil)}$. If there is an even number of data points, some software estimates the median as the average of $x_{(\frac{n}{2})}$ and $x_{(\frac{n}{2}+1)}$.

Example 9.7 (– waiting for a bus – median)**.** *Estimate the median of the time spent waiting for the bus using the observed data:* 0.1 0.3 0.5 0.7 1.0 1.9 2.8 *3.4* 3.5 3.8 5.3 7.7 8.6 8.7 11.1

The median is estimated as the middle observed value. Therefore with a probability of about 50% we will be waiting for the bus for less than 3.4 minutes and also for more than 3.4 minutes.

9.3 Point estimators

9.3.1 Introduction

From the measured data we can estimate the real value of the parameter θ using a *point estimator*:

Definition 9.8. A *point estimator* of a parameter θ is a function $\hat{\theta}_n(X_1, \ldots, X_n)$ of the random sample which does not depend on θ.

Notes:

- A point estimator is an example of a statistic. A *statistic* is an arbitrary function of the random sample which does not depend on the parameter θ.
- Generally, we can also construct a point estimator of a function of a parameter $g(\theta)$.
- A typical example is $g(\lambda) = \dfrac{1}{\lambda} = \mathrm{E}\,X$ for the exponential distribution.

9.3.2 Most common point estimators

- *Sample mean* – point estimator of the expectation $\mathrm{E}\,X$:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

- *Sample variance* – point estimator the of variance $\operatorname{var} X$:

$$s_n^2 = s_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

- *Sample standard deviation* – point estimator of the standard deviation $\sqrt{\operatorname{var} X}$:

$$s_n = \sqrt{s_n^2}.$$

- *k-th sample moment* – point estimator of k-th moment $\mu_k = \mathrm{E}\,X^k$:

$$m_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k.$$

- *Sample covariance* – point estimator of the covariance $\operatorname{cov}(X,Y)$:

$$s_{X,Y} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n).$$

- *Sample correlation coefficient* – point estimator of the correlation coefficient $\rho(X,Y)$:

$$r_{X,Y} = r = \frac{s_{X,Y}}{s_X s_Y}\,,$$

  where s_X and s_Y are square roots of the sample variances of X and Y.

9.3.3 Properties of point estimators

A point estimator as a function of the random sample is itself also a random variable with some distribution which obviously depends on the parameter θ.

A "*good estimator*" $\hat{\theta}_n$ should be in some way *close* to the true *value* of θ for all values θ and for all realizations of the random sample from F_θ.

Usually we want an estimator to be *unbiased*:

Definition 9.9. An estimator $\hat{\theta}_n$ of the parameter θ is called *unbiased* if

$$\mathrm{E}\,\hat{\theta}_n(X_1,\dots,X_n) = \theta \qquad \text{for all } \theta \in \Theta.$$

Unbiasedness means that an estimator does not have a systematic error, e.g., that it does not produce systematically larger or smaller values.

The next desirable property of estimators is *consistency*:

Definition 9.10. An estimator $\hat{\theta}_n$ of the parameter θ is called *consistent* if for all $\theta \in \Theta$:

$$\hat{\theta}_n \xrightarrow{P} \theta \qquad \text{for} \quad n \to \infty.$$

In other words if for all $\varepsilon > 0$: $\mathrm{P}(|\hat{\theta}_n(X_1,\dots,X_n) - \theta| \geq \varepsilon) \to 0$. Consistency means that by choosing a large n, the error of the estimate will be sufficiently small.

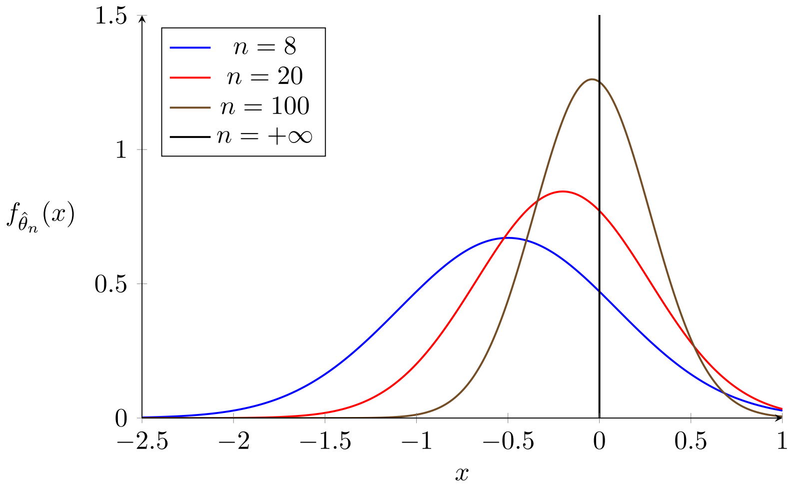
Theorem 9.11. *Let* $\mathrm{E}\,\hat{\theta}_n^2 < +\infty$ *for all* n*. If for* $n \to +\infty$ *it holds that*

$$\mathrm{E}\,\hat{\theta}_n \to \theta \quad \textit{and} \quad \operatorname{var}\hat{\theta}_n \to 0,$$

then $\hat{\theta}_n$ *is a* consistent *estimator.*

Proof. Proof can be found in bibliography. □

Convergence of the densities of a consistent estimator $\hat{\theta}_n$ with the true value of $\theta = 0$.



Example 9.12.

Sample mean

Consider a random sample $X_1, \dots, X_n$ from a distribution $F_{(\mu,\sigma^2)}$ where $\mathrm{E}\, X_i = \mu$ and $\operatorname{var} X_i = \sigma^2$.

- The sample mean $\bar{X}_n$ is *unbiased*:

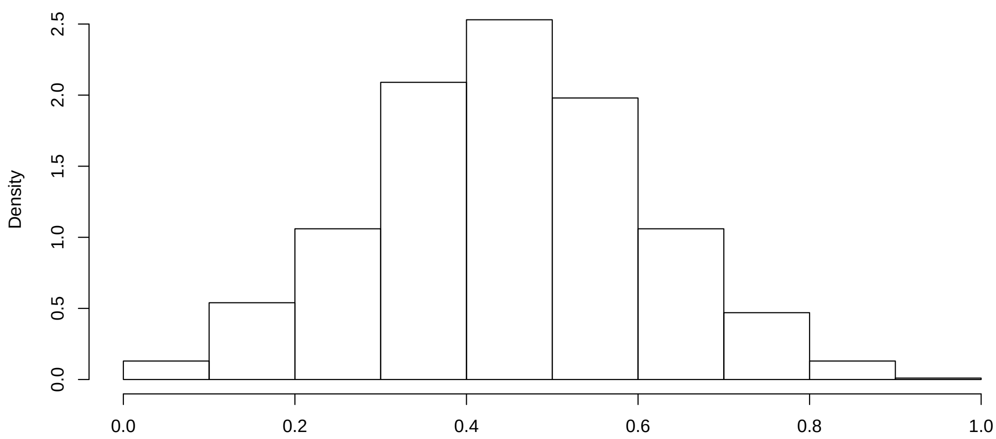
$$\mathrm{E}\, \bar{X}_n = \mathrm{E}\, \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{n}\, \mathrm{E} \sum_{i=1}^{n} X_i = \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\, X_i = \frac{n\mu}{n} = \mu.$$

- It is also *consistent*: from the weak law of large numbers we get that

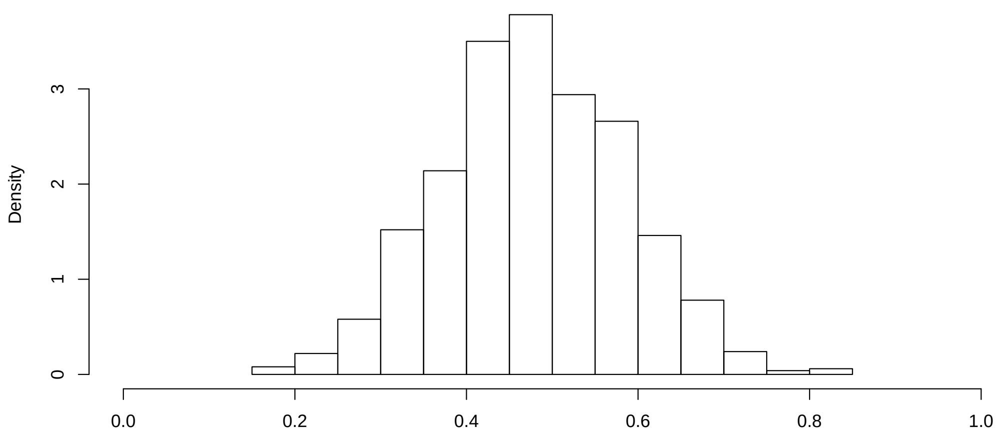
$$\bar{X}_n \xrightarrow{\mathrm{P}} \mu \quad \text{for } n \to \infty.$$

- The same follows from previous theorem and the fact that $\operatorname{var} \bar{X}_n = \dfrac{\sigma^2}{n} \to 0.$

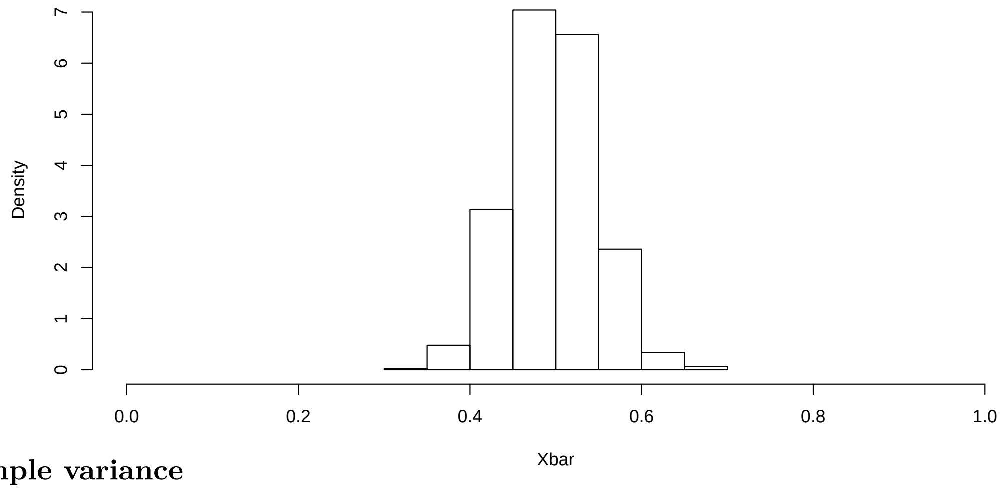
The *sample mean* $\bar{X}_n$ is thus an *unbiased* and *consistent* estimator of the expectation.
Histogram of the proportion of heads among 10 coin tosses (1000 simulations).



Histogram of the proportion of heads among 20 coin tosses (1000 simulations).



Histogram of the proportion of heads among 100 coin tosses (1000 simulations).



Sample variance

Consider a random sample $X_1, \dots, X_n$ from a distribution $F_{(\mu,\sigma^2)}$ where $\mathrm{E}\, X_i = \mu$ and $\operatorname{var} X_i = \sigma^2$. We want to estimate the variance σ^2 using the sample variance s_n^2. First we rewrite s_n^2 as

$$\begin{aligned} s_n^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \\ &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2\right) \\ &= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2 - 2\sum_{i=1}^{n}X_i\bar{X}_n + n\bar{X}_n^2\right) \\ &= \frac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2 - n\bar{X}_n^2\right). \end{aligned}$$

- *Unbiasedness*: since $\mathrm{E}\, X_i^2 = \sigma^2 + \mu^2$ and $\mathrm{E}\, \bar{X}_n^2 = \sigma^2/n + \mu^2$, we get

$$\begin{aligned} \mathrm{E}\, s_n^2 &= \frac{1}{n-1}\,\mathrm{E}\left(\sum_i X_i^2 - n\bar{X}_n^2\right) = \frac{1}{n-1}\left(n\,\mathrm{E}\, X_i^2 - n\,\mathrm{E}\, \bar{X}_n^2\right) \\ &= \frac{1}{n-1}\left(n\sigma^2 + n\mu^2 - n\sigma^2/n - n\mu^2\right) = \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2. \end{aligned}$$

  This is the reason why we divide by number $n-1$ instead of n, which can be more natural at the first glance. But such estimate would no longer be unbiased, it will be only asymptotically unbiased.

- *Consistency*: from the law of large numbers we get $\bar{X}_n \xrightarrow{n\to\infty} \mu = \mathrm{E}\, X_i$ and also $\frac{1}{n}\sum_i X_i^2 \xrightarrow{n\to\infty} \mathrm{E}\, X_i^2$. Thus we get

$$\begin{aligned} s_n^2 &= \frac{1}{n-1}\left(\sum_i X_i^2 - n\bar{X}_n^2\right) = \frac{n}{n-1}\left(\frac{1}{n}\sum_i X_i^2 - \bar{X}_n^2\right) \\ &\xrightarrow{n\to\infty} 1\cdot(\mathrm{E}\, X_i^2 - \mu^2) = \mathrm{E}\, X_i^2 - (\mathrm{E}\, X_i)^2 = \operatorname{var} X_i = \sigma^2. \end{aligned}$$

The *sample variance* s_n^2 is thus an *unbiased* and *consistent* estimator of the variance σ^2.

Quality of an unbiased estimator

Often we can construct several unbiased estimators of a given parameter. In this case we try to find the best of them, meaning the one with the smallest variance.

Definition 9.13. An estimator $\widehat{\theta}_n^{\text{best}}(X_1, \dots, X_n)$ is called the best unbiased estimator of the parameter θ if it is unbiased and for all other unbiased estimators $\widehat{\theta}_n$ of parameter θ it holds that

$$\operatorname{var}(\widehat{\theta}_n) \geq \operatorname{var}(\widehat{\theta}_n^{\text{best}}) \quad \text{for all } \theta \in \Theta$$

There exists a lower bound for the variance of an unbiased estimator (Rao - Cramer lower bound). If we find an unbiased estimator with the variance equal to this lower bound, we have the best unbiased estimator.

Theorem 9.14. *For binomial, Poisson, exponential, and normal distribution the sample mean is the best unbiased estimator of the expected value. For the normal distribution the sample variance is the best unbiased estimator of the variance.*

9.3.4 Method of moments

For a simple and quick (but sometimes not optimal) estimate of the parameters, the *method of moments* can be used. Let $X_1, \dots, X_n$ be a sample from a distribution with a d-dimensional parameter $\theta = (\theta_1, \dots, \theta_d)$.

Steps of the method of moments:

- Compute the theoretical moments $\operatorname{E} X_i^k$, for $k = 1, \dots, d$.
- Express the parameters as functions of the moments.
- Estimate the theoretical moments by their empirical versions:

$$\widehat{\operatorname{E} X_i^k} = m_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k.$$

- Insert the estimated moments and find the parameter estimates by solving the corresponding equations.

The method is useful because the law of large numbers implies that $m_k \to \operatorname{E} X_i^k$ for $n \to +\infty$. The estimates are thus always consistent.

Suppose $X_1, \dots, X_n$ form a random sample from a distribution $F_{(\mu,\sigma^2)}$ where $\operatorname{E} X_i = \mu$ and $\operatorname{var} X_i = \sigma^2$.

- The first two theoretical moments are

$$\operatorname{E} X_i = \mu, \qquad \operatorname{E} X_i^2 = \operatorname{var} X_i + (\operatorname{E} X_i)^2 = \sigma^2 + \mu^2.$$

- The parameters can be expressed as functions of the moments:

$$\mu = \operatorname{E} X_i, \qquad \sigma^2 = \operatorname{E} X_i^2 - (\operatorname{E} X_i)^2.$$

- We estimate $\mathrm{E}\,X_i$ using m_1 and $\mathrm{E}\,X_i^2$ using m_1 and m_2.
- The *estimators* of the expectation and variance are then

$$\hat{\mu}_n = \bar{X}_n \qquad \text{and} \qquad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2.$$

- After some algebra

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i^2 - 2X_i\bar{X}_n + (\bar{X}_n)^2\right) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{n-1}{n}s_n^2.$$

This estimator of the variance is consistent, but *not unbiased*. However, the extent of the bias will decrease, as $\frac{n-1}{n} \to 1$ for $n \to \infty$. Quite often a meaningful unbiased estimator does not exist and we must be satisfied with an asymptotically unbiased or biased estimator.

9.3.5 Maximum likelihood method

Example 9.15. *Suppose that among four coin tosses we obtained the sequence H,T,H,H. How can we estimate the expected proportion of Heads?*

X_1, X_2, X_3, X_4 form a random sample from the Bernoulli distribution with the parameter p with realizations $1, 0, 1, 1$. The probability of such realization is:

$$L(p) = \mathrm{P}(\text{H,T,H,H}) = \mathrm{P}(X_1 = 1 \cap X_2 = 0 \cap X_3 = 1 \cap X_4 = 1) = p^3(1-p).$$

As an estimate of the parameter p we take the value for which the obtained realization has the largest probability. Thus we find the maximum of the function $L(p)$.

$$\frac{\mathrm{d}L}{\mathrm{d}p}(p) = \frac{\mathrm{d}}{\mathrm{d}p}(p^3 - p^4) = 3p^2 - 4p^3 = p^2(3-4p) = 0.$$

Stationary points are 0 and $\frac{3}{4}$ and the maximum is achieved at point $\frac{3}{4}$. Hence we obtain the estimate $\widehat{p}_n = \frac{3}{4}$, which can be guessed from the set up.

Consistent estimators with desirable properties can be obtained using the *maximum likelihood* method. The aim is to *maximize* the *likelihood function* for given observations.

Definition 9.16. Let the random sample $X_1, \ldots, X_n$ have a distribution given by the joint density

$$f_\theta(\mathbf{x}) = \prod_{i=1}^{n} f_\theta(x_i) \quad \text{for a continuous distribution or}$$

$$p_\theta(\mathbf{x}) = \prod_{i=1}^{n} \mathrm{P}_\theta(X_i = x_i) \quad \text{for a discrete distribution.}$$

With values of $\mathbf{x} = (x_1, \ldots, x_n)$ fixed, the function $f_\theta(\mathbf{x})$, or $p_\theta(\mathbf{x})$, as a function of θ is called the *likelihood function* and is denoted as $L(\theta; \mathbf{x})$ or simply $L(\theta)$.

The likelihood function depends only on the parameter θ. The values $x_1, \ldots, x_n$ are treated as known and fixed.

Definition 9.17. The value $\hat{\theta}_n$ of the parameter θ maximizing the likelihood function $L(\theta; \mathbf{x})$ for a given random sample realization $\mathbf{X} = \mathbf{x}$ is called the *maximum likelihood estimator* (MLE) of the parameter θ. It means that

$$L(\hat{\theta}_n; \mathbf{x}) \geq L(\theta; \mathbf{x}) \quad \text{for all } \theta \in \Theta.$$

Notes:

- We can take $g(\hat{\theta}_n)$ as the maximum likelihood estimator of a function $g(\theta)$.
- Often it is advantageous to maximize the function $\ln L(\theta; \mathbf{x})$, because the logarithm turns a product into a sum.
- In the case of a k-dimensional parameter $\theta = (\theta_1, \ldots, \theta_k)$ we usually solve a system of equations
$$\frac{\partial \ln L(\theta_1, \ldots, \theta_k; \mathbf{x})}{\partial \theta_j} = 0 \quad \text{for } j = 1, \ldots, k.$$
- If certain *regularity conditions* are met (see literature), the maximum likelihood estimates are consistent, asymptotically unbiased and asymptotically normal.

Example 9.18 (– parameter of the exponential distribution)**.** Construct the MLE estimate of the parameter $\lambda > 0$ of the exponential distribution $\operatorname{Exp}(\lambda)$. The likelihood function for n observed values $x_1, \ldots, x_n$ (random sample realization) is clearly:

$$L(\lambda; \mathbf{x}) = f_\lambda(\mathbf{x}) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum x_i}.$$

In this case it is advantageous to maximize $\ln L(\lambda; \mathbf{x}) = n \ln(\lambda) - \lambda \sum_{i=1}^{n} x_i$.

After differentiating we obtain:

$$\frac{\mathrm{d} \ln L(\lambda; \mathbf{x})}{\mathrm{d}\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0.$$

A solution of this equation is the *maximal likelihood estimator* $\hat{\lambda}_n = \dfrac{n}{\sum_{i=1}^{n} x_i} = \dfrac{1}{\bar{x}_n}$.

Using the second derivative we can check that the obtained point is indeed the maximum.

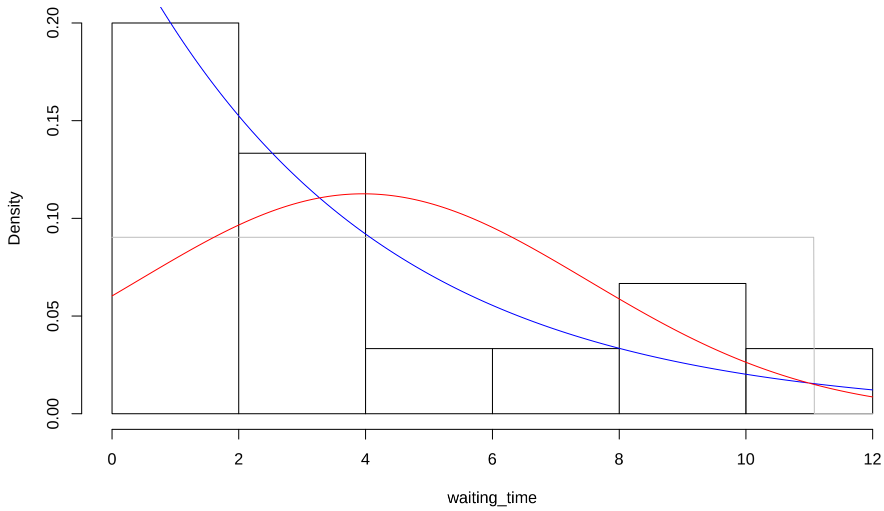
Example 9.19 (– waiting for a bus – comparison of distributions)**.** *Try fitting known continuous distributions on the observed waiting times from before. Estimate their parameters and compare the densities with the histogram.*

0.1	0.3	0.5	0.7	1.0	1.9	2.8	3.4	3.5	3.8	5.3	7.7	8.6	8.7	11.1

We try fitting the uniform $\operatorname{Unif}(0, b)$, exponential $\operatorname{Exp}(\lambda)$ and normal $\mathrm{N}(\mu, \sigma^2)$ distributions with estimated parameters:

Distribution	Estimated parameters	
Uniform	$a = 0$	$\hat{b}_n = \max(x_1, \ldots, x_{15}) \doteq 11.1$
Exponential	$\hat{\lambda}_n = \dfrac{1}{\bar{x}_n} \doteq 0.25$	–
Normal	$\hat{\mu}_n = \bar{x}_n \doteq 3.96$	$s_n^2 \doteq 12.56.$

Compare the histogram with the fitted densities.



The exponential distribution seems to provide the best fit.