Basic notions of probability

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Probability and Statistics

BIE-PST, WS 2023/24, Lecture 1

Content

- **Probability theory:**
  - Events, probability, conditional probability, Bayes' theorem, independence of events.
  - Random variables, distribution function, functions of random variables, characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values, covariance and correlation, important discrete and continuous distributions.
  - Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, conditional expected value.
  - Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.
- **Mathematical statistics:**
  - Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
  - Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Requirements for passing the course

- **Tutorials:**
  - there will be 6 small tests, each for 6 points, the 5 best results will count – 30p
  - homework assignment – 10p
  - needed at least 20p from 40p possible.
- **Exam:**
  - compulsory written exam max 60p – at least 30p needed
  - points from exam and tutorials will be added
  - optional theoretical exam – max 5p
  - taking the theoretical exam is possible only after successfully passing the written exam.

Bibliography

English books:

- D. P. Bertsekas & J. N. Tsitsiklis: **Introduction to Probability, Athena Scientific** MIT (2008)
- G. R. Grimmett & D. R. Stirzaker: **Probability and Random Processes**, Oxford University Press (2001)
- Ch. M. Grinstead & J. L. Snell: **Introduction to Probability**, AMS (1997) – (online)

Probability and statistics

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What do we understand as *random*? In real life we often encounter processes (experiments, tests, natural phenomena, . . .), for which we are not sure, how they will end and which result will occur.

Exact prediction may not be possible because they are either too complex or we do not have all necessary information available.

Usually we say that the result is unpredictable or **random** and is given by chance.

Probability theory vs mathematical statistics

Probability theory quantifies the unpredictability from a mathematical point of view.

- Outcomes of an experiment are assigned **probability**, giving the ideal proportion of cases when the outcome will occur.
- Starting from simple models, complex problems may be solved.
- e.g., if we know that a coin is balanced, we can compute what is the probability of getting $100\times$ Heads out of 1000 tosses.

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Mathematical statistics estimates the unpredictability using experimental data.

- Outcomes of a repeated experiment are used for estimation.
- Probabilistic models are suggested and verified.
- e.g., if we get only $100\times$ Heads out of 1000 tosses, is it enough evidence to say that a coin is not balanced?

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Imperfection of the definition:

- What to do if the die is unbalanced?
- What to do if there are infinitely many possible results?

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✓ Recall yourself the basic combinatorics!

Recap of Combinatorics

Consider a set of n elements, where $n \in \mathbb{N}$.

- The number of ways to order these elements (**permutations**) is $n!$
- The number of ways to select k elements without repetitions when the order is important (**variations**) is $\dfrac{n!}{(n-k)!}$.
- The number of ways to select k elements with repetitions when the order is important (**variations with repetition**) is n^k.
- The number of ways to select k elements without repetitions when the order is not important (**combinations**) is $\binom{n}{k}$.
- The number of ways to select k elements with repetitions when the order is not important (**combinantions with repetition**) is $\binom{n+k-1}{k}$.

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Imperfection of the definition:

- How to introduce unequal distribution of probability?
- What to do with objects of infinite size?

Geometric definition of probability

Example – Romeo and Juliet

Romeo and Juliet have to meet at a secret place between midday and 1 p.m. Each of them arrives at a random moment between midday and 1 p.m., but will wait for only 15 minutes. If the partner does not arrive, the waiting person will leave.

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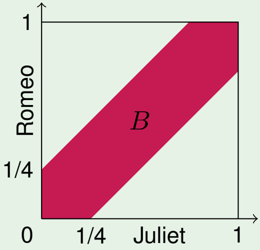

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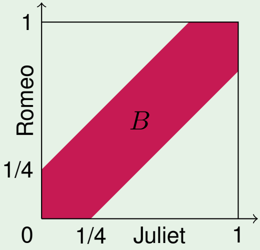

The total area is 1. The coloured area corresponds to the successful meeting.

$$\mathrm{P}(B) = \frac{1 - (3/4)\cdot(3/4)}{1} = \frac{7}{16}.$$

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The Sample space should be detailed enough to distinguish between different results but it should ignore unimportant details.

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- Height of the missile above the earth surface: $\Omega = [0, +\infty)$
- Random text in email in `UTF-32` encoding (constant length) of maximal length 1MB. The maximal number of characters in the message is

$$\frac{1\text{ MB}}{32\text{ bits}} = \frac{2^{20}\text{ bytes}}{4\text{ bytes}} = 2^{18} = 262144.$$

Thus: $\Omega = \{(x_1, x_2, \ldots, x_{2^{18}}) \,|\, x_i \in \mathbb{N}, 0 \leq x_i < 2^{32}, \text{ for all } i\}$

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- Throwing darts into $T \subset \mathbb{R}^2$: $\Omega = T \cup \{*\}$, where $\{*\}$ is a one point set representing the outcome "the dart does not reach the target".
If the target is divided into, say, 5 strips and we are interested only in which strip was reached: $\Omega = \{1, 2, 3, 4, 5, *\}$.

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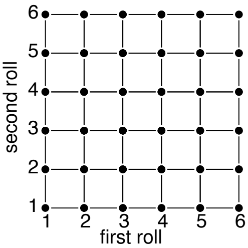
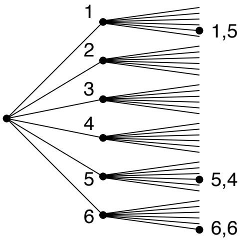
- Tossing a coin infinitely many times: countable space $\Omega = \{H, T\}^{\mathbb{N}}$.

Visualization of an outcome of a series of experiments

Two rolls of a die:
A **coordinate** description or a sketch in the form of a **tree** where each sequence of results of particular rolls corresponds to a single leaf that is uniquely determined by the path from the root to the leaf (in the illustration, only 3 leaves corresponding to 3 outcomes are explicitly marked).





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Even numbers are 2,4, and 6. The event A denoting that "an even number appears" is

$$A = \{2,4,6\} \subset \Omega.$$

Operations with events

It is possible to apply all classical set operations on events (events are sets). In probability theory, a specific terminology is used for this operations:

- A^c complement of A – **no outcome in A occurs**
- $A \cap B$ intersection of A and B – **both A and B occur**
- $A \cup B$ union A and B – **either A or B or both**
- $A \setminus B$ difference A and B – **A but not B**
- $A \subset B$ subset – **if A then B**
- $\emptyset$ empty set – **impossible event**
- Ω collection of objects – **whole sample space**
- ω member of Ω – **outcome, elementary event**

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Clearly: $\Omega \in \mathcal{F}$ – contains the sample space itself.

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Examples – possible σ-algebras

- $\mathcal{F} = \{\emptyset, \Omega\}$ is a σ-algebra.
- For an arbitrary $A \subset \Omega$, $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ is a σ-algebra.
- $\mathcal{F} = 2^{\Omega}$ – all subsets create a σ-algebra;
- Borel σ-algebra on $\mathbb{R}$ – smallest σ algebra containing all open intervals.

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ii) **normalization:** $\mathrm{P}(\Omega) = 1$,

iii) $\sigma-$**additivity:** if $A_1, A_2, \ldots \in \mathcal{F}$ is a collection of disjoint events (i.e., if $A_i \cap A_j = \emptyset$ for $\forall\, i, j$ with $i \neq j$), then

$$\mathrm{P}\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mathrm{P}(A_i).$$

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A **probability measure** P on $(\Omega, \mathcal{F})$ is a function $\mathrm{P} : \mathcal{F} \to \mathbb{R}$ satisfying:

i) **non-negativity:** for all $A \in \mathcal{F}$ it holds that $\mathrm{P}(A) \geq 0$

ii) **normalization:** $\mathrm{P}(\Omega) = 1$,

iii) **$\sigma-$additivity:** if $A_1, A_2, \ldots \in \mathcal{F}$ is a collection of disjoint events (i.e., if $A_i \cap A_j = \emptyset$ for $\forall\, i, j$ with $i \neq j$), then

$$\mathrm{P}\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mathrm{P}(A_i).$$

The triplet $(\Omega, \mathcal{F}, \mathrm{P})$ is called a **probability space**.

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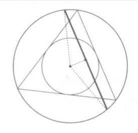
The choice of P determines what we understand as **“random”**. Vague assignment can lead to “paradoxes”.

Bertrand paradox

Example – Bertrand paradox (Joseph Bertrand, 1889)



What is the probability that a **randomly placed chord** χ on the unit circle will be longer than an the side ℓ of an equilateral triangle in the unit circle? I.e., what is $\mathrm{P}(A)$, where $A = \{|\chi| > \ell\}$.

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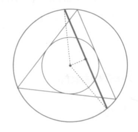

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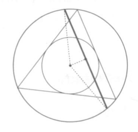

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Option 1: We choose randomly uniformly the **centre** of χ:

$$\Omega_1 = \{x \in \mathbb{R}^2 \,:\, |x| < 1\}, \quad A_1 = \{x \in \Omega_1 \,:\, |x| < 1/2\}, \quad \mathrm{P}_1(A_1) = \frac{\pi(1/2)^2}{\pi 1^2} = \frac{1}{4}.$$

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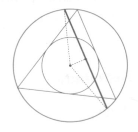

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Option 2: We choose randomly uniformly the **angle** and the direction (irrelevant thanks to the rotation symmetry) of chord χ observed from the circle centre:

$$\Omega_2 = (0, \pi], \quad A_2 = (2\pi/3, \pi], \quad \mathrm{P}_2(A_2) = \frac{\pi/3}{\pi} = \frac{1}{3}.$$

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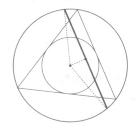

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Option 3: We choose randomly uniformly the **distance of the chord χ from the circle centre** and (again irrelevant) the direction:

$$\Omega_3 = [0, 1), \quad A_3 = [0, 1/2), \quad \mathrm{P}_3(A_3) = \frac{1}{2}.$$

Intermezzo

How to establish $\mathcal{F}$?

- Finite or countable Ω:
  - We can take $\mathcal{F}$ as all subsets of Ω. ($\mathcal{F} = 2^{\Omega}$, i.e., power set of Ω)
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  - Events are arbitrary subsets of Ω.

- For an uncountable Ω:
  - It is not possible to assign a positive probability to each $A \subset \Omega$, because then we would have $\mathrm{P}(\Omega) = \infty$.
  - If $\Omega \subset \mathbb{R}^d$ is some subinterval of $\mathbb{R}^d$ (e.g., $[0, +\infty)$) we can take $\mathcal{F}$ as the Borel σ-algebra on Ω.
  - Except for an at most a countable subset, each singular point must have a zero probability.

Definition of probability for “classical” settings

Definition of probability for uniform distribution of finite number of outcomes:

If Ω is finite with equally likely realizations:

$$\mathrm{P}(A) = \frac{\#A}{\#\Omega} = \frac{\text{number of favorable outcomes}}{\text{number of all outcomes}}.$$

Geometric definition of probability:

Let Ω be any arbitrary space with a measure μ, i.e., we can measure size (area, capacity, etc.), and $0 < \mu(\Omega) < +\infty$. For any event $A \subset \Omega$ we define:

$$\mathrm{P}(A) = \frac{\mu(A)}{\mu(\Omega)} = \frac{\text{size of } A}{\text{size of } \Omega}.$$

It can be easily verified that both approaches satisfy the formal definition of probability as stated above.

Properties of probability

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Consequences:

- $0 \leq \mathrm{P}(A) \leq 1$ – from **v)**
- $\mathrm{P}(A \cup B) \leq \mathrm{P}(A) + \mathrm{P}(B)$ – from **iv)**

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Proof

i) We create a sequence of disjoint events: $A_i = \emptyset$ for all $i \in \mathbb{N}$. From property **iii) of probability measure** we have

$$\mathrm{P}(\emptyset) = \mathrm{P}\left(\bigcup_{i=1}^{+\infty} \emptyset\right) = \sum_{i=1}^{+\infty} \mathrm{P}(\emptyset)\,,$$

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ii) We create sequence of disjoint events: $A_1 = A,\ A_2 = B$ and $A_i = \emptyset$ for $i > 2$. From properties **i)** and **iii)** of probability measure we have

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iii) $1 = \mathrm{P}(\Omega) = \mathrm{P}(A \cup A^c) = \mathrm{P}(A) + \mathrm{P}(A^c)$. Thus $\mathrm{P}(A^c) = 1 - \mathrm{P}(A)$.

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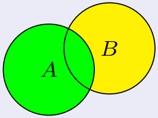
iv) The set $A \cup B$ can be written as the disjoint union $A \cup (B \setminus A)$.



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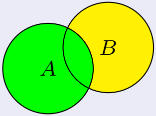

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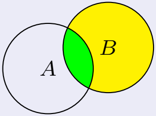

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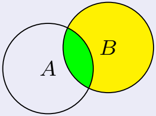
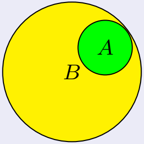


v) If $A \subset B$, then $A \cap B = A$.

$$\mathrm{P}(B) = \mathrm{P}(B \setminus A) + \mathrm{P}(A \cap B) = \mathrm{P}(B \setminus A) + \mathrm{P}(A) \geq \mathrm{P}(A)$$



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For 3 events:

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)$$

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i) The set $\bigcup_{i=1}^{+\infty} A_i$ can be written as a disjoint union $\bigcup_{i=1}^{+\infty} \left(A_i \setminus \bigcup_{k=1}^{i-1} A_k \right)$. From property **iii)** of probability measure we have

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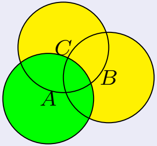

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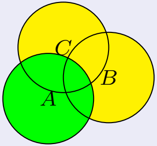

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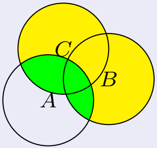

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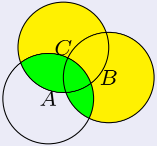

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and again, applying point **ii)** of the previous theorem to $\mathrm{P}(B \cup C)$ and to $\mathrm{P}((A \cap B) \cup (A \cap C))$

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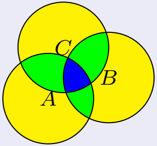
Proof

i) The set $\bigcup_{i=1}^{+\infty} A_i$ can be written as a disjoint union $\bigcup_{i=1}^{+\infty}\left(A_i \setminus \bigcup_{k=1}^{i-1} A_k\right)$. From property **iii)** of probability measure we have

$$\mathrm{P}\left(\bigcup_{i=1}^{+\infty} A_i\right) = \mathrm{P}\left(\bigcup_{i=1}^{+\infty}\left(A_i \setminus \bigcup_{k=1}^{i-1} A_k\right)\right) = \sum_{i=1}^{+\infty} \mathrm{P}\left(A_i \setminus \bigcup_{k=1}^{i-1} A_k\right) \leq \sum_{i=1}^{+\infty} \mathrm{P}(A_i).$$

ii) We prove the statement for 3 events A, B, C.



The set $A \cup B \cup C$ can be written as a disjoint union $A \cup ((B \cup C) \setminus A)$.
From point **ii)** of the previous theorem it follows that

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}((B \cap C) \setminus A).$$

Since $\mathrm{P}(B \cup C) = \mathrm{P}((B \cup C) \setminus A) + \mathrm{P}(A \cap (B \cup C))$, we have:

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B \cup C) - \mathrm{P}((A \cap B) \cup (A \cap C)).$$

and again, applying point **ii)** of the previous theorem to $\mathrm{P}(B \cup C)$ and to $\mathrm{P}((A \cap B) \cup (A \cap C))$ finally we have:

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C).$$

□

Continuity of probability

Theorem

Let $A_1, A_2, \ldots$ be a sequence of events ***increasing*** *in the sense of inclusion, i.e., such that $A_1 \subset A_2 \subset A_3 \subset \ldots$. If we denote*

$$A = \bigcup_{i=1}^{+\infty} A_i,$$

then $\mathrm{P}(A) = \lim_{n\to+\infty} \mathrm{P}(A_n)$.

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Similarly, let $B_1, B_2, \ldots$ be a sequence of events ***decreasing*** *in the sense of inclusion, i.e., such that $B_1 \supset B_2 \supset B_3 \supset \cdots$. If we denote*

$$B = \bigcap_{i=1}^{+\infty} B_i,$$

then $\mathrm{P}(B) = \lim_{n \to +\infty} \mathrm{P}(B_n)$.

Continuity of probability

Proof

We prove the first part of the statement:

The set A can be written as the disjoint union $A = \bigcup_{i=1}^{+\infty} (A_i \setminus A_{i-1})$

$\square$

Continuity of probability

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The set A can be written as the disjoint union $A = \bigcup_{i=1}^{+\infty} (A_i \setminus A_{i-1})$
Then it holds:

$$\mathrm{P}(A) = \mathrm{P}\left(\bigcup_{i=1}^{+\infty} (A_i \setminus A_{i-1})\right) = \sum_{i=1}^{+\infty} \mathrm{P}(A_i \setminus A_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{P}(A_i \setminus A_{i-1})$$
$$= \lim_{n\to\infty} \mathrm{P}(A_n)$$

□

Continuity of probability

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Then it holds:

$$P(A) = P\left(\bigcup_{i=1}^{+\infty} (A_i \setminus A_{i-1})\right) = \sum_{i=1}^{+\infty} P(A_i \setminus A_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i \setminus A_{i-1})$$
$$= \lim_{n\to\infty} P(A_n)$$

We prove the second part of the statement by means of the De Morgan rules and the proof of the first part:

$$P(B) = P\left(\bigcap_{i=1}^{+\infty} B_i\right) = P\left(\left(\bigcup_{i=1}^{+\infty} B_i^c\right)^c\right) = 1 - P\left(\bigcup_{i=1}^{+\infty} B_i^c\right)$$

□

Continuity of probability

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We prove the first part of the statement:

The set A can be written as the disjoint union $A = \bigcup_{i=1}^{+\infty} (A_i \setminus A_{i-1})$
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The sets B_i^C satisfy the assumptions of the first part of the Theorem and thus:

$$P(B) = 1 - \lim_{n\to\infty} P(B_n^c) = \lim_{n\to\infty} (1 - P(B_n^c)) = \lim_{n\to\infty} P(B_n).$$

□

Recap

A random experiment is represented using a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, where

- Ω is the set of possible results;
- $\mathcal{F}$ is a system of subsets of Ω;
- elements $A \in \mathcal{F}$ are called random events;
- the probability measure P is a function which assigns to the random events a real value from 0 to 1, representing the ideal proportion of cases in which the events occur.

If there is only a finite number of possible results with equal probabilities, then

$$\mathrm{P}(A) = \frac{|A|}{|\Omega|}.$$