

Random variables II.

(Characteristics of random variables)

Lecturer:
Francesco Dolce

Department of Applied Mathematics
Faculty of Information Technology
Czech Technical University in Prague

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Probability and Statistics

BIE-PST, WS 2023/24, Lecture 4



Content

- **Probability theory:**

- ▶ Events, probability, conditional probability, Bayes' Theorem, independence of events.
- ▶ Random variables, distribution function, functions of random variables, **characteristics of random variables: expected value, variance, moments, generating function, quantiles, critical values**, covariance and correlation, important discrete and continuous distributions.
- ▶ Random vectors, joint and marginal distributions, functions of random vectors, independence of random variables, conditional distribution, conditional expected value.
- ▶ Markov's and Chebyshev's inequality, weak law of large numbers, strong law of large numbers, Central limit theorem.

- **Mathematical statistics:**

- ▶ Point estimators, sample mean, sample variance, properties of point estimators, Maximum likelihood method.
- ▶ Interval estimators, hypothesis testing, one-sided vs. two-sided alternatives, linear regression, estimators of regression parameters, testing of linear model.

Recap

- A **random variable** X is a measurable function, which assigns real values to the outcomes of a random experiment.

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- The **distribution** of X gives the information of the probabilities of its values and is uniquely given by the **distribution function**:

$$F_X(x) = P(X \leq x).$$

- There are two major types of random variables:
 - ▶ **discrete**, taking only countably many possible values;
 - ▶ **continuous**, taking values from an interval.

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- The **distribution** of X gives the information of the probabilities of its values and is uniquely given by the **distribution function**:

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- There are two major types of random variables:
 - ▶ **discrete**, taking only countably many possible values;
 - ▶ **continuous**, taking values from an interval.
- The distribution can be given by:
 - ▶ for discrete distributions by the **probabilities** of possible values $P(X = x_k)$.
 - ▶ for continuous distributions by the **density** f_X for which

$$F_X(x) = \int_{-\infty}^x f(t) dt.$$

Expected value

One of the important characteristics of a random variable is its **expected value**.

Definition

The **expected value** (or **expectation** or **mean value**) of a discrete random variable X with values x_1, x_2, \dots , resp., of a continuous random variable X with density f_X , is given as

$$E X = \sum_k x_k P(X = x_k) \quad (\text{discrete})$$

resp., as

$$E X = \int_{-\infty}^{\infty} x f_X(x) dx, \quad (\text{continuous})$$

if the sum or the integral converges absolutely.

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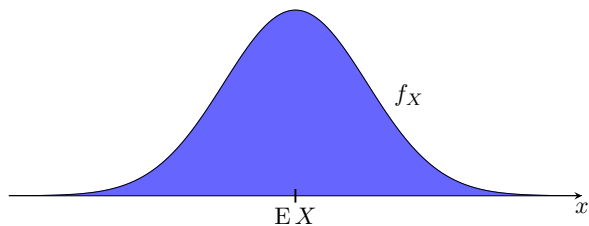
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From the definition it follows that $E X$ can be interpreted as the x coordinate of the **center of the mass** of the probability.

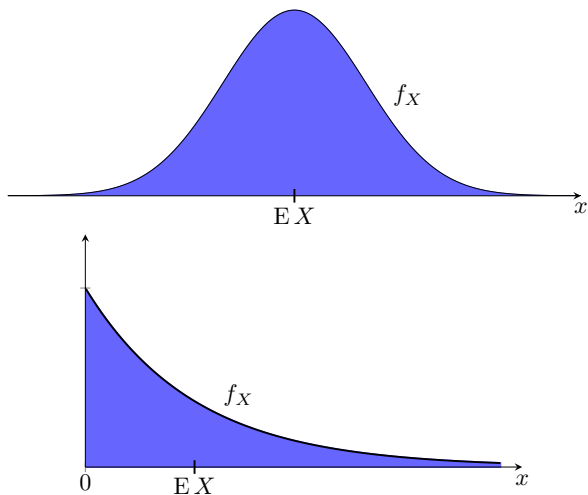
Visualization of the expectation

$E X$ is taken as the expected value of the next experiment or as the weighted average (mean) or the center of mass of all possible values.



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Example of the computation of the expectation

Example – tossing two coins

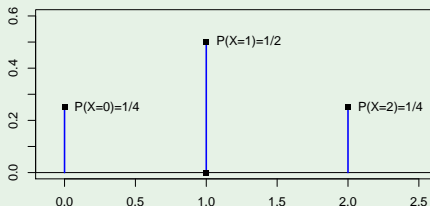
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There are four possible results, which are equally likely: $\Omega = \{TT, HT, TH, HH\}$. Therefore we can obtain 0, 1 or 2 Heads, with probabilities of $1/4$, $1/2$ and $1/4$, respectively.

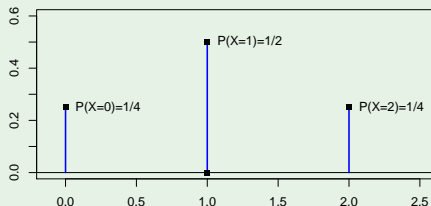


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There are four possible results, which are equally likely: $\Omega = \{TT, HT, TH, HH\}$. Therefore we can obtain 0, 1 or 2 Heads, with probabilities of $1/4$, $1/2$ and $1/4$, respectively.



The expectation is then computed as the probability-weighted average of the possible values:

$$E X = \sum_k x_k P(X = x_k) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{2}{4} = 1.$$

Example – discrete uniform distribution

Example – rolling a six-sided die

Suppose we roll a balanced six-sided die one time. Let X denote the number of points rolled. What is the expectation of X ?

Example – discrete uniform distribution

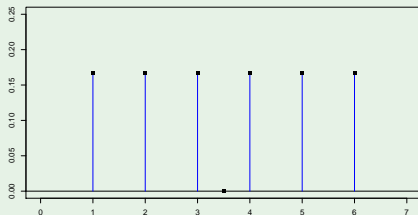
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Suppose we roll a balanced six-sided die one time. Let X denote the number of points rolled. What is the expectation of X ?

k	1	2	3	4	5	6
$P(X = k)$	1/6	1/6	1/6	1/6	1/6	1/6

The expectation is computed as the weighted average of possible results:

$$E X = \sum_{k=1}^6 k \cdot P(X = k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5.$$



Example – discrete non-uniform distribution

Example – rolling two six-sided dice

Suppose we roll two balanced six-sided dice and keep the larger result of the two. Let X denote the number of points rolled, meaning $X = \max(\text{roll 1}, \text{roll 2})$. What is the expectation of X ?

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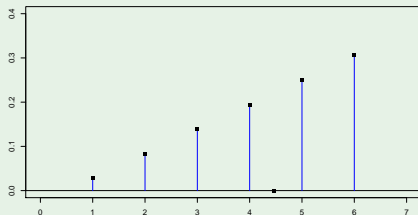
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k	1	2	3	4	5	6
$P(X = k)$	1/36	3/36	5/36	7/36	9/36	11/36

The expectation is computed as the weighted average of possible results:

$$E X = \sum_{k=1}^6 k \cdot P(X = k) = \frac{1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11}{36} = \frac{161}{36} \doteq 4.47.$$



Expected value of a function of a random variable

The expected value $E(g(X))$ of a function of a random variable can be computed without determining the distribution of the random variable $Y = g(X)$.

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Theorem

Let X and $Y = g(X)$ for a given function g be random variables.

i) If X has a **discrete** distribution, then

$$EY = E g(X) = \sum_{\text{all } x_k} g(x_k) P(X = x_k),$$

under the assumption that the sum converges absolutely.

ii) If X has a **continuous** distribution, then

$$EY = E g(X) = \int_{-\infty}^{\infty} g(x) f_X(x) dx,$$

if the integral converges absolutely.

Expected value of the function of a random variable

Proof

Suppose first that X is a discrete random variable. Denote the variable $Y = g(X)$ and its values y_1, y_2, \dots . Then

$$\begin{aligned} E(g(X)) &= EY = \sum_{\text{all } y_j} y_j P(Y = y_j) = \sum_{\text{all } y_j} y_j P(g(X) = y_j) \\ &= \sum_{\text{all } y_j} \left(y_j \sum_{x_k : g(x_k) = y_j} P(X = x_k) \right) = \sum_{\text{all } y_j} \sum_{x_k : g(x_k) = y_j} y_j P(X = x_k) \\ &= \sum_{\text{all } y_j} \sum_{x_k : g(x_k) = y_j} g(x_k) P(X = x_k) = \sum_{\text{all } x_k} g(x_k) P(X = x_k). \end{aligned}$$

The proof for continuous random variables is more difficult, we achieve it with the help of the following lemma only for function g taking non-negative values. □

Lemma

If X is a non-negative random variable with the distribution function F , then

$$E X = \int_0^{\infty} [1 - F(x)] dx = \int_0^{\infty} P(X > x) dx.$$

Expected value of the function of a random variable

Proof

Suppose that X is a continuous random variable and the function g takes only non-negative values. Then

$$E(g(X)) = EY = \int_0^{\infty} P(Y > y) dy$$



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Proof

Suppose that X is a continuous random variable and the function g takes only non-negative values. Then

$$E(g(X)) = EY = \int_0^{\infty} P(Y > y) dy = \int_0^{\infty} P(g(X) > y) dy$$



Expected value of the function of a random variable

Proof

Suppose that X is a continuous random variable and the function g takes only non-negative values. Then

$$\begin{aligned} E(g(X)) &= EY = \int_0^{\infty} P(Y > y) \, dy = \int_0^{\infty} P(g(X) > y) \, dy \\ &= \int_0^{\infty} \left(\int_{\{x: g(x) > y\}} f_X(x) \, dx \right) \, dy \end{aligned}$$



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$$\begin{aligned} E(g(X)) &= EY = \int_0^{\infty} P(Y > y) \, dy = \int_0^{\infty} P(g(X) > y) \, dy \\ \text{see (*)} \quad &= \int_0^{\infty} \left(\int_{\{x: g(x) > y\}} f_X(x) \, dx \right) \, dy = \iint_{\{(x,y): 0 < y < g(x)\}} f_X(x) \, d(x, y) \end{aligned}$$

(*) We used $P(X \in A) = \int_A f_X(x) \, dx$ for $A = \{x : g(x) > y\}$.



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Expected value of the function of a random variable

Proof

Suppose that X is a continuous random variable and the function g takes only non-negative values. Then

$$\begin{aligned} E(g(X)) &= EY = \int_0^{\infty} P(Y > y) dy = \int_0^{\infty} P(g(X) > y) dy \\ &= \int_0^{\infty} \left(\int_{\{x: g(x) > y\}} f_X(x) dx \right) dy = \iint_{\{(x,y): 0 < y < g(x)\}} f_X(x) d(x, y) \\ &= \int_{\{x: 0 < g(x)\}} \left(\int_0^{g(x)} f_X(x) dy \right) dx \quad (g(x) \text{ is non-negative}) \\ &= \int_{-\infty}^{\infty} f_X(x) \left(\int_0^{g(x)} dy \right) dx \end{aligned}$$



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 &= \int_{-\infty}^{\infty} f_X(x) \left(\int_0^{g(x)} dy \right) dx = \int_{-\infty}^{\infty} g(x) f_X(x) dx.
 \end{aligned}$$

(*) We used $P(X \in A) = \int_A f_X(x) dx$ for $A = \{x : g(x) > y\}$. If g is a general function we decompose it to its positive and negative parts which are both non-negative functions. Then we write $Eg(X) = EY = EY^+ - EY^- = Eg^+(X) - Eg^-(X)$ and use the above mentioned proof. \square

Properties of the expected value

For **computation**, the following properties of the expected value are important. Notice that these properties hold for the expectation of both discrete and continuous random variables.

Theorem

The expected value of a random variable X has the following properties:

- i) If $X \geq 0$, then $E(X) \geq 0$.*
- ii) If $a, b \in \mathbb{R}$, then $E(aX + b) = a E(X) + b$ (if $E X$ is finite).*
- iii) A constant random variable $X = c$ has expectation equal to the constant $E(X) = c$.*

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Notes:

- These properties of expectation do not depend on the type of random variable – discrete, continuous or mixed.
- For discrete, continuous or mixed random variables X and Y with finite expectations it holds that (we will prove it later)

$$E(aX + bY) = a E X + b E Y, \quad \forall a, b \in \mathbb{R}.$$

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These formulas can be used to simplify practical computing.

Properties of the expected value

Proof

- i) For a discrete non-negative random variable X it holds that $x_k P(X = x_k) \geq 0, \forall k$.

$$\text{Therefore } E(X) = \sum_{\text{all } x_k} x_k P(X = x_k) \geq 0.$$

For a continuous non-negative random variable X it holds that $f_X(x) = 0$ for $x < 0$.

$$\text{Therefore } E(X) = \int_0^{\infty} x f_X(x) dx \geq 0.$$

- ii) For a discrete random variable X it holds that

$$\begin{aligned} E(aX + b) &= \sum_{\text{all } x_k} (ax_k + b) P(X = x_k) \\ &= a \sum_{\text{all } x_k} x_k P(X = x_k) + b \sum_{\text{all } x_k} P(X = x_k) \\ &= a E(X) + b. \end{aligned}$$

For a continuous random variable X the proof is similar.

- iii) Consider $a = 0$ in ii).



Variance

Definition

The **variance** $\sigma^2 \equiv \text{var } X$ of a random variable X is defined as

$$\text{var } X = E(X - E X)^2.$$

The **standard deviation** of a random variable X is defined as

$$\text{s.d. } X = \sqrt{\text{var } X}.$$

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The following properties of the variance are useful for **practical computations**:

Theorem

For the variance it holds that:

- i) *For all $a, b \in \mathbb{R}$ and a random variable X it holds that*

$$\text{var}(aX + b) = a^2 \text{var } X.$$

- ii) *A constant random variable $X = c \in \mathbb{R}$ has zero variance ($\text{var } c = 0$).*

Variance

While computing the variance it is often tedious to calculate the sum of values $(x_i - \mathbb{E} X)^2 \mathbb{P}(X = x_i)$ or the integral of $(x - \mathbb{E} X)^2 f_X(x)$.

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$$\begin{aligned}\text{var}(X) &= \mathbb{E}((X - \mathbb{E} X)^2) = \mathbb{E}(X^2 - 2X(\mathbb{E} X) + (\mathbb{E} X)^2) \\ &= \mathbb{E}(X^2) - \mathbb{E}(2X(\mathbb{E} X)) + \mathbb{E}((\mathbb{E} X)^2) \\ &= \mathbb{E}(X^2) - 2(\mathbb{E} X)(\mathbb{E} X) + (\mathbb{E} X)^2 \\ &= \mathbb{E}(X^2) - (\mathbb{E} X)^2.\end{aligned}$$

We get the formula: $\text{var}(X) = \mathbb{E}((X - \mathbb{E} X)^2) = \mathbb{E}(X^2) - (\mathbb{E} X)^2$
or simply: $\text{var} X = \mathbb{E} X^2 - (\mathbb{E} X)^2$.

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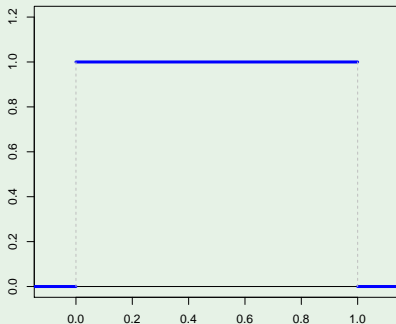
Notice that $\text{var}(X)$ is always non-negative (it is the expectation of a non-negative variable $(X - \mathbb{E} X)^2$). Therefore: $(\mathbb{E} X)^2 \leq \mathbb{E}(X^2)$.

Romeo, Juliet and the expectation and variance

Example – expectation and variance of the uniform distribution

Suppose that Romeo arrives at the meeting point according to the uniform distribution with the density:

$$f_X(x) = \begin{cases} 1 & \text{for } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$



What are the expectation and the variance of Romeo's arrival?

Romeo, Juliet and the expectation and variance

Example – expectation and variance of the uniform distribution

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Example – expectation and variance of the uniform distribution

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The expectation can be computed from the definition:

$$E X = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \cdot 1 dx = \left[\frac{1}{2} x^2 \right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}.$$

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The expectation of the square is computed similarly:

$$E X^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^1 x^2 1 dx = \left[\frac{1}{3} x^3 \right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

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The variance is obtained using the computational formula:

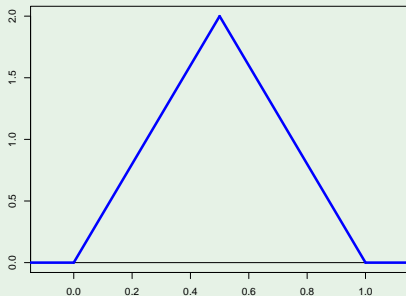
$$\text{var } X = E X^2 - (E X)^2 = 1/3 - (1/2)^2 = 4/12 - 3/12 = 1/12.$$

Romeo, Juliet and the expectation and variance

Example – expectation and variance of a non-uniform distribution

Suppose that Juliet arrives at the meeting point according to a non-uniform distribution with the density:

$$f_Y(y) = \begin{cases} 4y & \text{for } y \in [0, 1/2] \\ 4 - 4y & \text{for } y \in [1/2, 1] \\ 0 & \text{otherwise.} \end{cases}$$



What are the expectation and the variance of Juliet's arrival?

Romeo, Juliet and the expectation and variance

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What is the expectation and variance of Juliet's arrival?

The expectation can be computed from the definition:

$$E Y = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^{1/2} y(4y) dy + \int_{1/2}^1 y(4 - 4y) dy = \dots = \frac{1}{2}.$$

Romeo, Juliet and the expectation and variance

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The expectation of the square is computed similarly:

$$E Y^2 = \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_0^{1/2} y^2(4y) dy + \int_{1/2}^1 y^2(4 - 4y) dx = \dots = \frac{7}{24}.$$

The variance is obtained using the computational formula:

$$\text{var } Y = E Y^2 - (E Y)^2 = 7/24 - (1/2)^2 = 7/24 - 6/24 = 1/24.$$

The expectation is the same in both cases, but Romeo's arrivals have a twice larger variance than Juliet's.

Moments of random variables

Definition

For $k \in \mathbb{N}$ we define the k -th moment μ_k of a random variable X as

$$\mu_k = E(X^k) = \begin{cases} \sum_{\text{all } x_i} x_i^k P(X = x_i) & \text{discrete} \\ \int_{-\infty}^{\infty} x^k f_X(x) dx & \text{continuous.} \end{cases}$$

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Notation: usually we write $E X^k$ instead of $E(X^k)$ and $E(X - \mu_1)^k$ instead of $E((X - \mu_1)^k)$.

Moments, expectation, variance, standard deviation

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Remark

Note that the variance is quadratic and therefore is measured in the units of X **squared**. The standard deviation is the square root of the variance and is therefore measured in the same units as X . This will be useful later.

Skewness

The **measure of asymmetry** around the mean is called **skewness**:

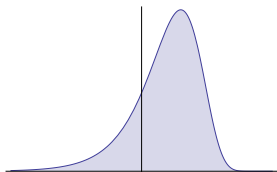
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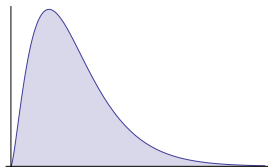
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Measure of asymmetry: for a unimodal density the coefficient γ_1 is **negative** if the **left** tail is longer and **positive** if the **right** tail is longer. It tells us to which side from the expected value is the bulk skewed:



$$\gamma_1 = -1.14$$



$$\gamma_1 = 1.26$$

Kurtosis

The **measure of “peakedness”** is called **(excess) kurtosis**:

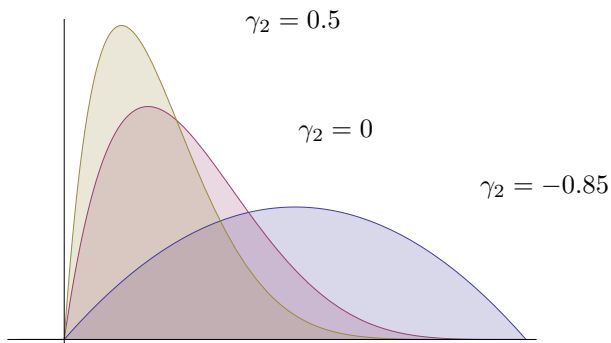
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This characteristics compares the shape (“peakedness”) of the density with the normal distribution:



Quantiles

Important characteristics of random variables used especially in statistics are the **quantiles** and the **critical values**:

Definition

Let X be a random variable with distribution function F_X and let $\alpha \in (0, 1)$. The point q_α is called the **α -quantile** of the variable X if

$$q_\alpha = \inf\{x | F_X(x) \geq \alpha\}.$$

Quantiles treated as a function of α are called the **quantile function** and are denoted as $F_X^{-1}(\alpha)$.

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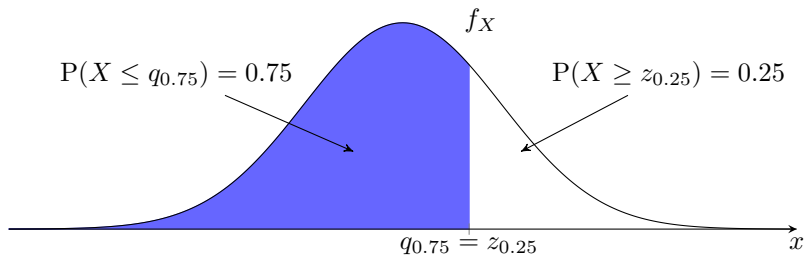
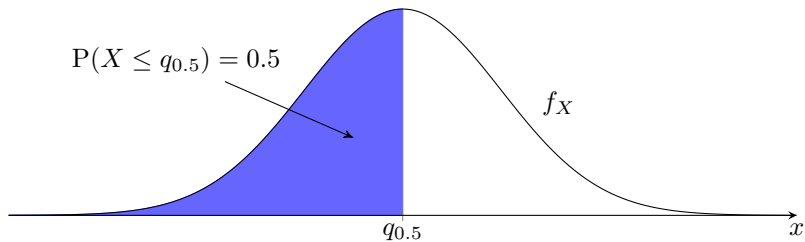
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- For F_X strictly increasing and continuous, the quantile function $F_X^{-1}(\alpha)$ is the inverse of F_X in the classic sense.
- For some particular distributions, special notation is used, e.g., the quantiles of the Gaussian distribution are denoted as u_α and the critical values as z_α .

Quantiles of the standard normal distribution



Romeo, Juliet and quantiles

Example – quantiles of the uniform distribution

Suppose that Romeo arrives at the meeting point according to the uniform distribution on the interval $[0, 1]$. Find the 5% and 95% quantiles of his arrival.

Romeo, Juliet and quantiles

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Suppose that Romeo arrives at the meeting point according to the uniform distribution on the interval $[0, 1]$. Find the 5% and 95% quantiles of his arrival.

The distribution function is found by integrating the density. We are interested in the region, where the density is positive – the interval $[0, 1]$:

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_0^x 1 dt = [t]_0^x = x.$$

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The distribution function is monotone, thus we can easily find the quantile function as its inverse:

$$F_X(q_\alpha) = \alpha \quad \Rightarrow \quad q_\alpha = \alpha \quad \Rightarrow \quad F_X^{-1}(\alpha) = \alpha.$$

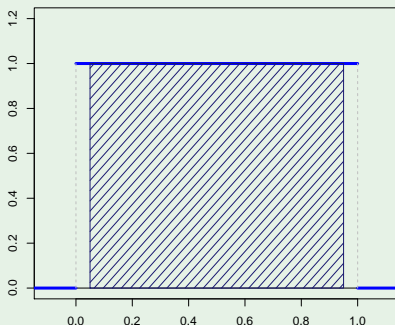
Therefore the quantiles are:

$$q_{0.05} = 0.05 = 3 \text{ min.} \quad \text{and} \quad q_{0.95} = 0.95 = 57 \text{ min.}$$

Romeo, Juliet and quantiles

Example – quantiles of the uniform distribution

With a 90% probability, Romeo arrives between the 3rd minute and the 57th minute.



Romeo, Juliet and quantiles

Example – quantiles of a non-uniform distribution

Suppose that Juliet arrives at the meeting point according to the non-uniform distribution with the triangular density from above. Find the 5% and 95% quantiles of her arrival.

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For $y \in [0, 1/2]$:

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For $y \in [1/2, 1]$:

$$F_Y(y) = \int_0^{1/2} 4t dt + \int_{1/2}^y (4 - 4t) dt = 1/2 + [4t - 2t^2]_{1/2}^y = 4y - 2y^2 - 1 = 1 - 2(y - 1)^2.$$

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The quantile function is found as the inverse of the distribution function:

$$F_Y(q_{0.05}) = 0.05 \Leftrightarrow 2q_{0.05}^2 = 0.05 \Leftrightarrow q_{0.05} = \sqrt{0.05/2} \doteq 0.16 = 9.5 \text{ min.}$$

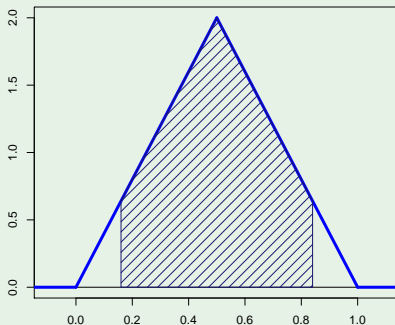
Similarly:

$$F_Y(q_{0.95}) = 0.95 \Leftrightarrow 1 - 2(q_{0.95} - 1)^2 = 0.95 \Leftrightarrow q_{0.95} = 1 - \sqrt{0.05/2} \doteq 0.84 = 50.5 \text{ min.}$$

Romeo, Juliet and quantiles

Example – quantiles of a non-uniform distribution

With a 90% probability, Juliet arrives between the 9.5th minute and the 50.5th minute.



The central interval denoting the time, between which the person arrives with a 90% probability, is considerably shorter for Juliet than for Romeo. This is in accordance with Juliet's arrival having a smaller variance.

Important quantiles

Quantiles divide the population into groups according to probabilities. The important dividing points are called:

- $q_{0.5}$ – **median**,
- $q_{0.25}$ – **lower quartile**,
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These quantiles can give us an overview of the variable in question:

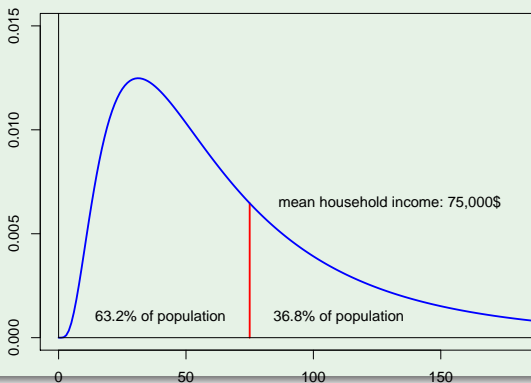
- The **median** provides a measure of **location** as an alternative to the expectation.
- The **interquartile range** $q_{0.75} - q_{0.25}$ provides a measure of **dispersion** as an alternative to the variance.

The expectation can sometimes differ from the median significantly. Especially for one-sided heavy-tailed distributions.

Expectation vs. median

Example – U.S. household incomes

According to the U.S Census Bureau, the mean yearly household income in 2014 was \$75,000. But 63.2% of population had lower incomes. The median income was \$56,000.



Moment generating function

Definition

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Theorem

For a random variable X with a generating function $M(s)$ it holds that:

$$\mathbb{E}(X^n) = \frac{d^n}{ds^n} M(s) \Big|_{s=0}.$$

Examples of generating functions

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$$\text{Thus } \text{var}(X) = (\lambda)^2 - (\lambda + \lambda^2) = \lambda.$$

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$$\frac{d^2}{ds^2} \frac{\lambda}{\lambda - s} = \frac{2\lambda}{(\lambda - s)^3} \implies E(X^2) = \frac{2}{\lambda^2} \quad \text{and} \quad \text{var}(X) = \frac{1}{\lambda^2}.$$

Recap

The **expectation** of a random variable X gives us its **center of mass** or the **expected average outcome**.

- For discrete random variables it is the average of its possible values weighted by their probabilities:

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The **variance** of a random variable X gives us the expected quadratic distance of the random variable from its expectation. It is defined as

$$\text{var } X = E(X - E X)^2$$

and can be computed as:

$$\text{var } X = E X^2 - (E X)^2.$$