Review - Basics of probability

The classic definition of probability

- Ω is a set of all possible outcomes of a random trial.
- $\omega \in \Omega$ is an elementary event.
- $A \subset \Omega$ is random event.
- Let Ω contain a **finite** number of elements: $\Omega = \{\omega_1, \dots, \omega_n\}$. Suppose that all elementary events are **equally likely**. Then we define the probability of a random event A as

$$\mathrm{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{n},$$

where $|A|$ is the number of elements of the set A (favorable events).

Geometric probability

- The state space Ω is identified as a geometric shape.
- The points lying inside correspond to elementary events having the same weight.
- The random events correspond to its subsets.
- The probability of event A is defined as the fraction of areas (volumes, etc):

$$\mathrm{P}(A) = \frac{|A|}{|\Omega|}$$

Properties

- $0 \leq \mathrm{P}(A) \leq 1$,
- $\mathrm{P}(A^c) = 1 - P(A)$,
- if $A \subset B$, then $\mathrm{P}(A) \leq \mathrm{P}(B)$ and $\mathrm{P}(B \setminus A) = \mathrm{P}(B) - \mathrm{P}(A)$,
- $\mathrm{P}(A \cap B) = \mathrm{P}(A) - \mathrm{P}(A \cap B^c)$,
- $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$,
- (principle of inclusion and exclusion):

$$\mathrm{P}(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{i=1}^{n} \mathrm{P}(A_i) - \sum\sum_{i<j} \mathrm{P}(A_i \cap A_j) + \dots + (-1)^{n+1}\, \mathrm{P}(A_1 \cap A_2 \cap \dots \cap A_n).$$

Combinatorics

- Permutation: number of all ordered n-tuples of n elements is $n!$.
- Variation: number of all ordered k-tuples of n elements is $n!/(n-k)!$.
- Combination: number of all unordered k-tuples from n elements is $\binom{n}{k}$ (choose k from n).

Exercises 1 - Basics of probability

1. Permutation, variation and combination

a) How many different records of the sequence $1, 2, 3, 4, 5$ exist?

b) How many possibilities are there to choose three people winning first, second and third price out of 20 people?

c) How many different outcomes can we get by choosing 3 out of 10 different balls?

d) How many different records of the sequence $1, 2, 3, 4, 5$ exist, such that 2 goes right after 1?

e) * How many different records of the sequence $1, 2, 3, 4, 5$ exist, such that 2 appears after (not necessarly immediatly after) 1?

2. Two people are repeatedly playing a fair game (probability of winning in each round is 0.5 for each player). The winner of a round gains one point. The player who first gains 6 points is the winner of the game. Both players bet the same amount of money and the winner takes the whole sum. However, the game is interrupted when player A has 5 points and player B has 3 points. How should they divide the money?
Hint: Compute the probability that player B would win at the end.

3. Two people are playing a game with 2 coins. If the result of the tosses is (H, T) or (T, H), player A wins. If the result is (H, H) or (T, T), player B wins. Count the probability that player A wins if

a) the probability of H is 0.5;

b) the probability of H is $p \in (0, 1)$. For which value of $p = \mathrm{P}(H)$ has player A the highest probability of winning?

4. Two people want to play a fair game, but they only have an unbalanced coin (with an unknown probability of Heads/Tails). Find a way to establish such a game.
Hint: Toss twice.

5. We are rolling twice a balanced die (probability of each outcome is $\frac{1}{6}$). What is the probability that the second rolled number is lower by one than the first?

6. We are rolling two balanced dice (probability of each outcome is $\frac{1}{6}$).

a) What is the probability that the sum of the results is 4?

b) How does the result change if the dice are unbalanced with $\mathrm{P}(3) = \frac{2}{7}$ and $\mathrm{P}(1) = \mathrm{P}(2) = \mathrm{P}(4) = \mathrm{P}(5) = \mathrm{P}(6) = \frac{1}{7}$?

7. A student has to choose exactly two of out of three voluntary courses: Mathematics, French and Literature. We know that the student chooses:

a) Literature with probability $\frac{5}{8}$,

b) French with probability $\frac{5}{8}$,

c) Literature and French with probability $\frac{1}{4}$.

What is the probability that the student chooses Mathematics? And what is the probability of Literature or Mathematics?

8. Prove that for arbitrary events A and B it holds that:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B).$$

9. Let A and B be events with probabilities $\mathrm{P}(A) = \frac{3}{4}$ and $\mathrm{P}(B) = \frac{1}{3}$. Show that $\frac{1}{12} \leq \mathrm{P}(A \cap B) \leq \frac{1}{3}$.

10. Consider the following game: A player has to choose from three doors. One door leads to a prize, two of the doors contain nothing. The player picks one door, but does not open it yet. The host of the game then opens one of the remaining two doors, which contains nothing. The player is presented with a choice of either staying with his original choice, or switching to the other remaining closed door. Which of the choices yields a better probability of getting the prize?

Reference:
`https://math.andyou.com/tools/montyhallsimulator/montysim.htm`
`https://en.wikipedia.org/wiki/Monty_Hall_problem`

Additional exercises - Basics of probability

11. We throw four six-sided dice. Find the probability that:

a) four different numbers appear;

b) only odd numbers appear;

c) the sum of numbers on all dice will be equal to 6;

d) the sum of numbers will be larger than 5;

e) at least one six appears.

12. There are 6 bottles of water and 4 bottles of vodka on a shelf, indistinguishable from each other. We randomly pick three bottles and taste them. Find the probability, that

a) exactly one of chosen bottles contained vodka;

b) at least one of chosen bottles contained vodka.

13. ($\star$) There are n different letters and n different envelopes, each with an address written on them. An absent-minded secretary places the letters in the envelopes at random.

a) What is the probability of at least one of the letters ending in the correct envelope?

b) Compute the limit of that probability for $n \to \infty$ and compare it with the exact results for $n = 5$ and $n = 10$.

Maxwell-Boltzmann scheme

14. ($\star$) There is a train having n wagons and r passengers are standing on a platform. Each passenger chooses one wagon at random and gets in. Suppose that the wagons are infinitely big.

a) Find the probability that there are exactly k passengers in the first wagon.

b) What is the probability of at least one passenger being in each wagon?

c) Compute the limit of the probability (a) for $n \to \infty$ and $r \to \infty$, so that $r/n \to \lambda > 0$.

Bose-Einstein scheme

15. ($\star$) The grandmother puts r pieces of candy into n boxes for her n grandchildren. The candies are distributed at random, with each state having the same probability

a) Find the probability that Alice gets exactly k pieces of candy.

b) Find the probability that each grandchild gets at least one piece of candy.

c) Compute the limit of the probability (a) for $n \to \infty$ and $r \to \infty$, so that $r/n \to \lambda > 0$.

16. ($\star$) There are n different keys on a ring, only one of which fits in the door in front of us. We are trying to open the door by trying the keys one after another at random and putting the wrong keys aside. What is the probability of opening the lock on the k-th try?

Geometric probability

17. We draw two independent real random numbers x and y from the interval $[0, 1]$. Compute the probability, that

a) $x + y \leq 1$.

b) $x^2 + y^2 \leq 1$.

18. Two friends agreed to meet between 1pm and 2pm on a secret place. Each of them comes at a random time within this time period. Each of them waits 20 minutes for the other one and then leaves. What is the probability that the two friends meet?

19. ($\star$) We cut a one meter long pole randomly at two points. What is the probability that it is possible to construct a triangle from the resulting three pieces?