Review - Conditional probability

Conditional probability:
Suppose A, B are random events with $\mathrm{P}(B) > 0$. The **conditional probability** of the event A given B is defined as

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Law of total probability:
Let $A, B_1, B_2, \ldots$ be random events such that $B_i \cap B_j = \emptyset$ for all $i \neq j$, $\bigcup_i B_i = \Omega$ and $\mathrm{P}(B_i) > 0$ for all $i = 1, 2, \ldots$. Then

$$\mathrm{P}(A) = \sum_i \mathrm{P}(A \cap B_i) = \sum_i \mathrm{P}(A|B_i)\,\mathrm{P}(B_i).$$

Bayes' Theorem:
Let $A, B_1, B_2, \ldots$ be random events such that $B_i \cap B_j = \emptyset$ for all $i \neq j$, $\bigcup_i B_i = \Omega$, $\mathrm{P}(B_i) > 0$ for all $i = 1, 2, \ldots$, and let $\mathrm{P}(A) > 0$. Then

$$\mathrm{P}(B_i|A) = \frac{\mathrm{P}(B_i \cap A)}{\mathrm{P}(A)} = \frac{\mathrm{P}(A|B_i)\,\mathrm{P}(B_i)}{\sum_j \mathrm{P}(A|B_j)\,\mathrm{P}(B_j)}.$$

Independence:
Two random events A, B are independent, if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B).$$

The random events $A_1, \ldots, A_n$ are independent, if for all $k \leq n$ and each subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$:

$$\mathrm{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathrm{P}(A_{i_1}) \cdot \ldots \mathrm{P}(A_{i_k}).$$

(Meaning that the multiplicative property has to be verified for all pairs, triples, ...etc.)

Exercises 1 - Conditional probability

1. A die is rolled twice. What is the probability that the sum of the rolls is larger than 7, given that the first outcome was 4?

2. Suppose that boys and girls are born independently and evenly.

a) What is the probability that a family with two children has two boys given that at least one is a boy?

b) How does the probability change if we know that the younger one is a boy?

c) How does the probability change if we know that at least one of the children is a boy born on Monday?

3. In a class, 70% of students are boys. 10% of boys play football and 5% of girls play football too. How many percent of students play football?

4. Two urns contain indistinguishable blue and white balls. In the first urn there are 2 white and 3 blue balls. In the second urn there are 3 white and 4 blue balls. We pick up randomly one ball from the first urn and insert it to the second urn (we do not look at it). Then we pick up randomly one ball from the second urn.

a) What is the probability that the second ball is blue?

b) If we pick up a blue ball from the second urn, what is the probability that a white ball was picked up from the first urn in the first place?

5. 90% of produced processors are fully functional. Our customer requires the percentage to be at least 99%. That is why the produced processors are tested. A case study shows that the test evaluates 80% of functional processors as acceptable and 10% of faulty processors as acceptable.

a) What is the probability that a produced processor is evaluated as acceptable?

b) Do we meet the requirements of our customer after the test? (**Hint:** compute the probability that the processor is completely functional if the test evaluates it as acceptable.)

c) Production of one processor costs 2$ and the test costs 0.2$. Processors which are evaluated as not acceptable are thrown away. For what price should the processors be sold to cover the expenses?

6. Analysis of our email account shows that 30% of delivered messages are spam. 65% of spam messages include the word "copy". In 15% of non spam messages the word "copy" is included too. Consider a newly delivered message including the word "copy". What is the probability that the message is spam?

7. Sensitivity of a test for a certain illness (i.e., the probability of the test being positive if the patient is ill) is 95%. The specificity of the test (i.e., the probability of the test being negative if the patient is not ill) is 97%. The examined illness affects 5% of population.

a) What is the probability that the patient has the illness given that the test is positive?

b) What is the probability that the patient has the illness given that the test is negative?

8. We have a pack of 52 cards (4 suits, 13 ranks).

a) Are the event of drawing a queen and the event of drawing hearts independent?

b) Does something change if we add one joker to the pack?

9. Let A and B be two independent events. Prove that A^c and B are independent, too. Further, prove that A^c and B^c are independent as well.

10. Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be a sample space where each outcome has the same probability. Denote the events $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 5\}$ and $C = \{1, 6, 7, 8\}$. Are these events independent? Are they at least pairwise independent?

11. Suppose there are three towns, A, B and C. There are two roads from A to B and two roads from B and C. In winter, each of the roads may get blocked by snowfall with probability p, independently on one another. What is the probability that we can get from A to C?

12 (Galton paradox)**.** Someone tosses three coins and tells you that *at least two* coins are showing the same results (two Heads or two Tails). You know that it is equally likely ($\frac{1}{2}$) to have Head or Tail on the third coin. Thus the probability of three equal results should be $\frac{1}{2}$. Do you agree? If not, where is the problem and what is the correct result?

Additional exercises - Conditional probability

Conditional probability

13. A friend throws a 6-sided die and covers the result. What is the probability of him rolling a 6 if the friend tells us (truthfully) that he has rolled an even number?

14. There are three boxes, one containing two gold coins, one with two silver coins and one containing one gold and one silver. We randomly select a box and randomly pick one coin. Suppose that it is gold. What is the probability of this box containing a second gold coin?

Law of total probability

15. A tennis player has a 60% success rate on the first service and a 80% success rate on the second service. What is the probability of a double fault?

16. We have two shipments of fruit in a dark storage room, 100 boxes from Spain and 300 boxes from Greece. There are 1% of spoiled boxes among the Spanish fruit and 5% among the Greek. We pick a box at random. What is the probability that it contains spoiled fruit?

17. (**Polya urn scheme**) An urn contains a white and b black balls. We draw randomly one ball, note its color, put it back and add d balls of the same color into the box. Then we draw again. What is the probability of picking a white ball on the second draw?

18. We roll two 6-sided dice over and over again. What is the probability of getting a sum of 5 sooner in the sequence than a sum of 7?

Bayes' Theorem

19. A factory producing microchips has found out that only 87% of the produced chips have acceptable quality, the others are faulty. A quality control test has been implemented, but it is not accurate. Only 80% of the good chips pass the test and only 85% of the faulty chips actually fail the test. Only chips that have passed the test are put on market. What is the probability that a chip that passed the test is actually good?

20. A certain course at school was so hard and convoluted, that only 5% of the students understood it. The final exam of the subject was somewhat luck-based, meaning that a student with good understanding passed the exam with 95% probability, and a student who did not understand passed with 25% probability. Suppose we meet a student who passed the test. What is the probability that he actually understood the subject, given that he passed the test?

Independence

21. Suppose A and B are disjoint events. Can these events be independent?

22. We throw two 6-sided dice, blue and green. Denote the events

- A: "even number on blue dice",
- B: "odd number on green dice",
- C: "the sum is odd".

Are the events A, B, C pairwise independent? Are the events A, B, C independent?

23. Suppose we throw three independent symmetric coins, with the results marked as H: "heads" and T: "tails". Consider the events

a) $A = \{HTT, HHT, HTH, HHH\}$,

b) $B = \{THH, TTH, THT, HHH\}$,

c) $C = \{HTT, HHT, TTT, HHH\}$.

Are the events A, B, C independent?

24. Alice, Bob and Celia are playing a game involving taking turns in flipping a symmetric coin. Whoever lands Heads, wins the game. If Tails occur, the coin is passed to the next player. Alice begins, then Bob, then Celia then again Alice etc. What is the probability of winning for each player?