Review - Random variables I

Random variables:

A **random variable** X is a measurable mapping (function) from the space $(\Omega, \mathcal{A})$ into $(\mathbb{R}, \mathcal{B})$. We assign real numbers $X(\omega)$ to elements $\omega \in \Omega$.

The **distribution** of the random variable X:

- describes the probabilities $\mathrm{P}(X \in B) = \mathrm{P}(\{\omega : X(\omega) \in B\})$ for all sets $B \in \mathcal{B}$,
- and is uniquely identified by the **distribution function** defined as

$$F(x) = \mathrm{P}(X \leq x) \qquad\qquad x \in \mathbb{R}.$$

Discrete random variables:

If the random variable X can have only **countable many** values $x_1, x_2, \dots$, we say that it has a **discrete distribution**.

- The distribution of X is characterised by the probabilities $p_k = \mathrm{P}(X = x_k)$, with $k = 1, 2, \dots$ for which $\sum_k p_k = 1$.
- The **distribution function** is partwise constant, with jumps of the size p_k at the points x_k.

Continuous random variables:

Suppose that for the random variable X with a distribution function F exists such a function $f \geq 0$ such that

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$$

Then we say that X has a **continuous distribution**. The function f is then called the **density** of X.

- A continuous random variable can have **uncountably many** values from a subinterval of $\mathbb{R}$.
- The distribution of X is characterised by the density $f \geq 0$. For each $B \in \mathcal{B}$ we have

$$\mathrm{P}(X \in B) = \int_B f(x)\,\mathrm{d}x.$$

Properties

a) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \mathrm{P}(X \in \mathbb{R}) = 1.$

b) The distribution function F is continuous on $\mathbb{R}$ and can be computed as

$$F(x) = \mathrm{P}(X \leq x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t.$$

c) For any $a \in \mathbb{R}$ we have $\mathrm{P}(X = a) = \int_{\{a\}} f(t)\,\mathrm{d}t = 0.$

d) If $a < b$, then

$$\mathrm{P}(a < X < b) = \mathrm{P}(a < X \leq b) = F(b) - F(a) = \int_a^b f(t)\,\mathrm{d}t.$$

Exercises 3 - Random variables

1. Consider the random variable X denoting the total number of Heads in two tosses.

a) Plot the distribution function of this random variable if the coin is balanced.

b) Do the same under assumption that the coin is not balanced, i.e., $\mathrm{P}(\text{“Heads”}) = p$.

c) Find the probabilities of values for both above mentioned cases.

2. We are rolling a six-sided dice until a 6 occurs. Denote the moment when a 6 occurs for the first time as the X-th attempt.

a) What is the distribution of the random variable X?

b) Find $\mathrm{P}(X > 3)$.

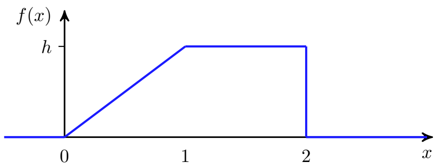
c) Find $\mathrm{P}(X > 7|X > 3)$.

3. Let F be a distribution function of random variable X. Prove that:

a) $\mathrm{P}(X > x) = 1 - F(x)$,

b) $\mathrm{P}(x < X \leq y) = F(y) - F(x)$.

4. Let X be a continuous random variable with the following density:



a) Write the density $f(x)$ analytically.

b) Determine the constant h.

c) Find the distribution function of the random variable X.

5. Consider the function

$$F(x) = \begin{cases} \frac{x^2}{2} & \text{for } x \in [0,1], \\ 2x - \frac{x^2}{2} - 1 & \text{for } x \in [1,2], \\ 0 & \text{elsewhere.} \end{cases}$$

a) Is the function $F(x)$ a distribution function of a continuous random variable? If not, is it possible to fix it easily?

b) Find the density of the (corrected) function.

c) Find the probability $\mathrm{P}\left(X \in (\frac{1}{2}, \frac{3}{2})\right)$.

6. Let X be a random variable with the distribution function $F_X(x)$. Consider the random variable $Y = aX + b$, where $a > 0$ and $b \in \mathbb{R}$. Find the distribution function of the random variable Y in terms of F_X.

7. * Let X be a continuous random variable with density:

$$f_X(x) = \begin{cases} 1 & \text{for } x \in (0,1), \\ 0 & \text{elsewhere.} \end{cases}$$

Consider the random variable $Y = \cos(\pi X)$. Find the distribution function and the density of Y.

8. * Let $\lambda > 0$. Consider a continuous random variable X with the density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \in (0, \infty), \\ 0 & \text{elsewhere.} \end{cases}$$

Denote F the distribution function of the random variable X. Consider the random variable $Y = F(X)$. Find the distribution function of Y.

9. Find the 0.9-quantile of the random variable X with distribution function $F_X(x) = 1 - e^{-2x}$.

Additional exercises - Random variables I

Discrete random variables

10. There are 8 keys on a ring. We are trying to open a door, for which only one key on the ring works. Because it's dark, we are selecting the keys at random. After each unsuccessful try, the key ring falls on the floor, we pick it up and try again until we unlock the door.

a) What is the probability distribution of the number of unsuccessful attempts before unlocking the door?

b) What is the probability of needing at most 6 unsuccessful attempts?

c) What is the probability of needing at least 10 unsuccessful attempts total, given that we already tried to open the door 6 times?

Continuous random variables

11. The length of a baby's afternoon sleep (in hours) is a random variable X with a uniform distribution on the interval $[0, 3]$, meaning that it has the density

$$f(x) = \begin{cases} \frac{1}{3} & x \in [0, 3], \\ 0 & \text{else.} \end{cases}$$

a) Compute the distribution function F of the random variable X and draw its graph.

b) What is the probability that the baby will sleep for exactly one hour?

c) What is the probability that it will sleep for at least one hour?

d) What is the probability that the length of its sleep will be between 30 minutes and two hours?

e) We know that the baby is already sleeping for one hour. What is the probability that the length of the whole sleep will be longer than two hours?

12. (Cauchy distribution) * Consider a random variable X with the density

$$f(x) = c \cdot \frac{1}{1 + x^2}.$$

Find the constant $c > 0$, so that f truly forms a density of X.