Review - Random vectors I: independence, conditionals

Joint distribution:
If $(X,Y)^T$ has a discrete distribution with values (x_i, y_j), the joint distribution is given by the probabilities $\mathrm{P}(X = x_i \cap Y = y_j)$ for all possible (x_i, y_j).

If $(X,Y)^T$ is continuous with a density $f(x,y)$, then the joint distribution is given by the joint density $f_{XY}(x,y)$.

Marginal distribution:
If $(X,Y)^T$ has a discrete distribution with values (x_i, y_j), then the marginal distribution of X is discrete and can be computed as

$$\mathrm{P}(X = x_i) = \sum_j \mathrm{P}(X = x_i \cap Y = y_j).$$

If $(X,Y)^T$ is continuous with a density $f(x,y)$, then the marginal density of X is computed as

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy.$$

Similarly for the marginal distribution of Y.

Independence:
Two random variables X and Y are independent, if

$$\mathrm{P}(X \leq x \cap Y \leq y) = \mathrm{P}(X \leq x)\,\mathrm{P}(Y \leq y) \quad \forall (x,y) \in \mathbb{R}^2.$$

If $(X,Y)^T$ has a continuous distribution then X and Y are independent if and only if $f_{XY}(x,y) = f_X(x)f_Y(y)$ for almost all $(x,y) \in \mathbb{R}^2$.

If $(X,Y)^T$ has a discrete distribution then X and Y are independent if and only if $\mathrm{P}(X = x_i \cap Y = y_j) = \mathrm{P}(X = x_i)\,\mathrm{P}(Y = y_j)$ for all possible (x_i, y_j).

Conditional distribution:

a) If $(X,Y)^T$ has a discrete distribution with values (x_i, y_j), then the conditional distribution of X given $Y = y$ is discrete and can be computed as

$$\mathrm{P}(X = x_i | Y = y_j) = \frac{\mathrm{P}(X = x_i \cap Y = y_j)}{\mathrm{P}(Y = y_j)}.$$

b) If $(X,Y)^T$ is continuous with a density $f(x,y)$, then the conditional density of X given $Y = y$ is computed as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Conditional expectation:
If $(X,Y)^T$ has a discrete distribution with values (x_i, y_j), then the conditional expectation of X given $Y = y$ is defined as

$$\mathrm{E}(X|Y = y_j) = \sum_{x_i} x_i\, \mathrm{P}(X = x_i | Y = y_j).$$

If $(X,Y)^T$ has a continuous distribution, then the conditional expectation of X given $Y = y$ is defined as

$$\mathrm{E}(X|Y = y) = \int_{-\infty}^{+\infty} x \cdot f_{X|Y}(x|y)\,\mathrm{d}x.$$

Similarly for the conditional expectation of Y given X.

EXERCISES 6 - RANDOM VECTORS I: INDEPENDENCE, CONDITIONALS

1. Suppose that a random vector (X, Y) has the following joint probability distribution:

$\mathrm{P}(X = x \cap Y = y)$		Y = 0	Y = 1	Y = 2
X	1	0.4	0.15	0.05
	2	0.3	0.06	0.04

a) Find the marginal distributions of the random variables X and Y.

b) Are X and Y independent?

c) Find the conditional distribution of $X|Y = y$.

d) Find $\mathrm{E}(X|Y = y)$.

e) Find the conditional distribution of $X + Y$ given $Y = y$.

2. Consider two continuous random variables X and Y with joint density

$$f(x, y) = \begin{cases} \frac{1}{5}(4xy + 4x - 8y) & \text{for } 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are X and Y independent?

b) Find $\mathrm{E}(X + Y)$.

c) Find the conditional density $f_{Y|X}(y|x)$.

d) Find $\mathrm{E}(Y|X = x)$.

3. Consider two continuous random variables X and Y with the joint density

$$f(x, y) = \begin{cases} \frac{1}{2} y e^{(-x-y^2)/2} & \text{for } x > 0 \text{ and } y > 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are X and Y independent?

b) Find $\mathrm{E}(X + Y)$.

4. Let X and Y be two continuous random variables with the joint density

$$f(x, y) = \begin{cases} \frac{1}{3}(-4xy - 4x + 8y + 8) & \text{for } 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Are X and Y independent?

b) Find $\mathrm{E}(X + Y)$.

5. Let X and Y be continuous random variables with the joint density

$$f(x, y) = \begin{cases} \frac{3}{8} x y^2 & \text{for } 0 \leq x \leq 1, \text{ and } -2 \leq y \leq 2\,, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the marginal density of the random variable X.

b) Find the conditional density $f_{Y|X}(y|x)$.

6. Suppose we roll two dice. Suppose that the random variable X is defined as the sum of both rolled numbers and the variable Y is defined as the difference between the number rolled on the first and the number rolled on the second dice. (i.e., $Y = 1^{st} - 2^{nd}$). Find $\mathrm{E}(X|Y = 2)$.

Additional exercises - Random vectors I: independence, conditionals

7. Two people are betting on the result of one roll of a six sided die. If a person is correct in his bet, he wins 1. If he is not correct, he loses 1. Suppose that person X bets on even numbers and person Y bets on large numbers $(4, 5, 6)$. Let X denote a random variable marking the winnings/losings of X after one game, same for Y.

a) Find the joint distribution of X and Y.

b) Find the marginal distributions for X and Y.

c) Are X and Y independent?

d) Find the conditional distribution of X given $Y = 1$ and given $Y = -1$.

8. Suppose that on on Halloween night you decide to take a walk in a haunted graveyard. Both vampires and zombies inhabit this ghostly place, and the more you stay the more probable an unwanted encouter will be. The probability of meeting a vampire (in hours) is denoted as V and the one of meeting a zombie as Z. Suppose that V and Z are continuous with the joint density

$$f_{VZ}(v, z) = \begin{cases} \frac{1}{2}e^{-v-z/2} & \text{for } v > 0,\ z > 0, \\ 0 & \text{else.} \end{cases}$$

a) What are the marginal distributions of V and Z?

b) Are V and Z independent?

c) At what time do you expect to encounter a vampire? And a zombie?