Review - Limit theorems

Chebyshev's inequality
Suppose that X is a random variable with a finite variance. Then for any $\varepsilon > 0$:

$$\mathrm{P}(|X - \mathrm{E}\,X| \geq \varepsilon) \leq \frac{\operatorname{var} X}{\varepsilon^2}.$$

Strong law of large numbers
Suppose that $X_1, X_2, \ldots$ are i.i.d. with $\mathrm{E}\,X_1 = \mu$, for which $\mathrm{E}\,|X_1| < \infty$. Then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mu.$$

Central limit theorem
Suppose that $X_1, X_2, \ldots$ are i.i.d. with $0 < \operatorname{var} X_1 < \infty$. Denote $S_n = \sum_{i=1}^{n} X_i$. Take

$$Z_n = \frac{S_n - n\,\mathrm{E}\,X_1}{\sqrt{n \operatorname{var} X_1}}.$$

Then for all $z \in \mathbb{R}$:

$$\mathrm{P}(Z_n \leq z) \xrightarrow{n\to\infty} \Phi(z),$$

where $\Phi(z)$ is the distribution function of the standardised normal distribution $\mathrm{N}(0,1)$. Equivalently we can write

$$Z_n = \frac{\overline{X}_n - \mathrm{E}\,X_1}{\sqrt{\operatorname{var} X_1}}\sqrt{n},$$

using means instead of sums.

Distribution function values and quantiles of $\mathrm{N}(0,1)$

z	0.000	0.050	0.100	0.150	0.200	0.250	0.300	0.350	0.400	0.450
$\Phi(z)$	0.500	0.520	0.540	0.560	0.579	0.599	0.618	0.637	0.655	0.674
z	0.500	0.550	0.600	0.650	0.700	0.750	0.800	0.850	0.900	0.950
$\Phi(z)$	0.691	0.709	0.726	0.742	0.758	0.773	0.788	0.802	0.816	0.829
z	1.000	1.050	1.100	1.150	1.200	1.250	1.300	1.350	1.400	1.450
$\Phi(z)$	0.841	0.853	0.864	0.875	0.885	0.894	0.903	0.911	0.919	0.926
z	1.500	1.550	1.600	1.650	1.700	1.750	1.800	1.850	1.900	1.950
$\Phi(z)$	0.933	0.939	0.945	0.951	0.955	0.960	0.964	0.968	0.971	0.974
z	2.000	2.050	2.100	2.150	2.200	2.250	2.300	2.350	2.400	2.450
$\Phi(z)$	0.977	0.980	0.982	0.984	0.986	0.988	0.989	0.991	0.992	0.993
z	2.500	2.550	2.600	2.650	2.700	2.750	2.800	2.850	2.900	2.950
$\Phi(z)$	0.994	0.995	0.995	0.996	0.997	0.997	0.997	0.998	0.998	0.998

α	0.5	0.8	0.9	0.95	0.975	0.99	0.995
$\Phi^{-1}(\alpha)$	0	0.842	1.282	1.645	1.960	2.326	2.576

Exercises 8 - Limit theorems

1. Consider n variables $X_1, \ldots, X_n$ with a common expected value $\mathrm{E}\, X_i = \mu$ and a common variance $\operatorname{var} X_i = \sigma^2$ for all $i \in \{1, 2, \ldots, n\}$. Suppose that these variables are non-correlated.

a) Show that $\mathrm{E}\, \bar{X}_n = \mu$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

b) Show that $\operatorname{var} \bar{X}_n = \frac{\sigma^2}{n}$.

To solve the following exercises we need to use the Central Limit Theorem.

2. Suppose that $X_1, \ldots, X_{64}$ are independent and identically distributed random variables with $\mathrm{E}\, X_i = 6$ and $\operatorname{var} X_i = 4$ for all $i \in \{1, \ldots, 64\}$.

a) Find the probability $\mathrm{P}\left(\sum_{i=1}^{64} X_i \leq 360\right)$.

b) Find the probability $\mathrm{P}\left(\bar{X}_n > 6.5\right)$.

c) Find the probability $\mathrm{P}\left(5.2 < \bar{X}_n < 6.9\right)$.

d) Find a constant a such that $\mathrm{P}\left(\bar{X}_n < a\right) = 0.36$.

e) Find a constant a such that $\mathrm{P}\left(6 - a < \bar{X}_n < 6 + a\right) = 0.73$.

3. The maximum load of a ship is 5000 kg. The weight of one passenger is a random variable with expected value 70 kg and standard deviation (square root of variance) 20 kg. Suppose that the passengers' weights are independent. How many passengers can embark so that the probability of overload is less than 0.001?

4. What is the approximate probability that in 1000 independent tosses of a fair coin we get 450-550 times Heads?

5. Before the election, there are 52% voters in favor of the Stark party and 48% voters favoring the Lannister party. What is the probability that a poll of size n=1500 respondents incorrectly shows the dominance of the Lannister party?

Additional exercises - Limit theorems

Chebyshev's inequality

6. We operate a river boat which has a maximum load of 6 tons and a maximum designed capacity of 50 passengers. Suppose that the weight of a passenger is a random variable with expectation $\mu = 80$ kg and variance $\sigma^2 = 20^2$ kg^2. Use the Chebyshev's inequality to construct an upper bound for the probability that the ship will sink with $n = 50$ independent passengers on board.

Central limit theorem

7. Let us consider the river boat of the previous exercise. Use the central limit theorem to compute the probability that the ship will sink with $n = 50$ independent passengers on board.

8. Suppose we have 100 independent light bulbs in storage. The time length until a light bulb breaks down follows the exponential distribution with $\lambda = 1/10\ h^{-1}$. When a light bulb breaks down, we replace it immediately with a new one. What is the probability that all the bulbs will operate for more than 1100 hours until we run through all of them?

9. A FTP server has 100 users, with each user having stored on average 1200 MB of data with a standard deviation of 400 MB. How large should be the hard drive of the server, so that it won't get full with a probability of 99%?

10. We make a series of tosses with a symmetric coin and take notes on the proportion of Heads. How many times should we toss the coin, so that the proportion of Heads will be between 0.4 and 0.6 with a 95% probability?

11. An insurance company has made a contract with 1000 customers. Each customer pays 1200 CZK. If a customer dies within the upcoming year, the company pays 80000 CZK to his relatives. The probability of a customer dying within a year is 0.01. What is the probability that the insurance company will come to a loss?