BIE-DML - Discrete Mathematics and Logic

Tutorial 2

Logical Laws, DNF, CNF

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2.1 Introduction

2.1.1 Logical Laws to remember – all in one table

Theorem 2.1. *Given any formula A, B, C, tautology $\top$ and contradiction $\bot$, the following logical equivalences hold:*

1. Commutative laws:	$A \wedge B \boxminus B \wedge A$	$A \vee B \boxminus B \vee A$
2. Associative laws:	$(A \wedge B) \wedge C \boxminus A \wedge (B \wedge C)$	$(A \vee B) \vee C \boxminus A \vee (B \vee C)$
3. Distributive laws:	$A \wedge (B \vee C) \boxminus (A \wedge B) \vee (A \wedge C)$	$A \vee (B \wedge C) \boxminus (A \vee B) \wedge (A \vee C)$
4. Identity laws:	$A \wedge \top \boxminus A$	$A \vee \bot \boxminus A$
5. Law of excluded middle:	$A \vee \neg A \boxminus \top$	
6. Law of non-contradiction:	$A \wedge \neg A \boxminus \bot$	$\neg(A \wedge \neg A) \boxminus \top$
7. Law of double negation:	$\neg\neg A \boxminus A$	
8. Idempotent laws:	$A \wedge A \boxminus A$	$A \vee A \boxminus A$
9. Universal bound laws:	$A \wedge \bot \boxminus \bot$	$A \vee \top \boxminus \top$
10. Negation of $\top$ and $\bot$:	$\neg\top \boxminus \bot$	$\neg\bot \boxminus \top$
11. De Morgan laws:	$\neg(A \wedge B) \boxminus \neg A \vee \neg B$	$\neg(A \vee B) \boxminus \neg A \wedge \neg B$
12. Absorption laws:	$A \vee (A \wedge B) \boxminus A$	$A \wedge (A \vee B) \boxminus A$
13. ”Golden rule”/”Silver rule”:	$A \Rightarrow B \boxminus \neg A \vee B$	$\neg(A \Rightarrow B) \boxminus A \wedge \neg B$

Remark 2.2. Given any formula A, B, tautology $\top$ and contradiction $\bot$, the table below presents the list of the most used logically equivalent formulas to tautology, contradiction, implication and equivalence:

1. $\top$	$A \vee \neg A$
	$A \Leftrightarrow A$
	$\neg(A \wedge \neg A)$
2. $\bot$	$A \wedge \neg A$
	$A \Leftrightarrow \neg A$
	$\neg(A \vee \neg A)$
3. $A \implies B$	$\neg A \vee B$
	$\neg(A \wedge \neg B)$
	$\neg B \Rightarrow \neg A$
4. $A \Leftrightarrow B$	$(A \Rightarrow B) \wedge (B \Rightarrow A)$
	$(A \Rightarrow B) \wedge (\neg A \Rightarrow \neg B)$
	$(A \vee \neg B) \wedge (\neg A \vee B)$
	$(A \wedge B) \vee (\neg A \wedge \neg B)$
	$\neg A \Leftrightarrow \neg B$

Lemma 2.3 (Properties of Implication)**.**

- $A \Rightarrow B \quad \not\models\!\!\!\!\dashv \quad B \Rightarrow A$
- $A \Rightarrow (B \Rightarrow C) \quad \not\models\!\!\!\!\dashv \quad (A \Rightarrow B) \Rightarrow C$

Lemma 2.4 (Properties of Equivalence)**.**

- $A \Leftrightarrow B \models\!\!\dashv B \Leftrightarrow A$ . . . *commutative law*
- $A \Leftrightarrow (B \Leftrightarrow C) \models\!\!\dashv (A \Leftrightarrow B) \Leftrightarrow C$ . . . *associative law*
- $\neg(A \Leftrightarrow B) \models\!\!\dashv \neg A \Leftrightarrow B$ . . . *negation*
- $A \Leftrightarrow A \models\!\!\dashv \top$
- $A \Leftrightarrow \neg A \models\!\!\dashv \bot$
- $A \Leftrightarrow \top \models\!\!\dashv A$
- $A \Leftrightarrow \bot \models\!\!\dashv \neg A$

2.1.2 Functionally Complete Systems of Logical Connectives

Definition 2.5 (Complete Systems)**.**
A set of logical connectives is (functionally) **complete** iff for any formula there is a logically equivalent formula containing connectives only from this set.

Definition 2.6 (Sheffer symbol and Peirce arrow)**.**
We define the **Sheffer symbol** $\uparrow$ (or NAND) as

$$A \uparrow B \models\!\!\dashv \neg(A \wedge B)$$

We define the **Peirce arrow** $\downarrow$ (or NOR) as

$$A \downarrow B \models\!\!\dashv \neg(A \vee B).$$

Theorem 2.7. *The following sets of connectives are functionally complete.*

1. $\{\neg, \vee\}$;
2. $\{\neg, \wedge\}$;
3. $\{\neg, \Rightarrow\}$;
4. *all sets containing the sets above;*
5. $\{\uparrow\}$;
6. $\{\downarrow\}$.

2.1.3 Disjunctive Normal Form and Conjunctive Normal Form of Formulas of PL

Definition 2.8 (Literals, Implicants, DNF)**.**

- A **literal** is an elementary formula or its negation.
- An **implicant** is a literal or a conjunction of literals.
- A formula is in **disjunctive normal form** (DNF) if it is an implicant or a disjuction of implicants.

Definition 2.9 (Clauses, CNF)**.**

- A **clause** is a literal or a disjunction of literals.
- A formula is in **conjunctive normal form** (CNF) if it is a clause or a conjunction of clauses.

Example 2.10.

- A; $\neg A$; B . . . literals
- $A \wedge \neg B$; $\neg A \wedge C \wedge B$; $\neg C$ . . . implicants
- $(A \wedge B \wedge C) \vee (\neg A \wedge \neg C)$ . . . DNF
- $(A \wedge \neg B) \vee (\neg A \wedge C \wedge B) \vee \neg C$ . . . DNF
- A; $\neg A$; $A \vee B$; $A \wedge \neg B$ (!!!) . . . DNF
- $A \vee \neg B$; $\neg A \vee C \vee B$; $\neg C$ . . . clauses
- $(A \vee \neg B) \wedge C$ . . . CNF
- $(A \vee \neg B) \wedge (\neg A \vee C) \wedge \neg C$ . . . CNF
- A; $\neg A$; $A \wedge \neg B$; $A \vee \neg B$ (!!!) . . . both DNF and CNF!

Theorem 2.11. *For every formula there is a logically equivalent formula which is in DNF and a logically equivalent formula which is in CNF.*

Remark 2.12. DNF and CNF are constructed by application of logical laws, mostly these ones:

- We use distributive laws (in both directions!) for transition between CNF and DNF.
  - $(G \wedge \neg H) \vee (\neg I \wedge J \wedge K) \models\!\dashv (G \vee \neg I) \wedge (G \vee J) \wedge (G \vee K) \wedge (\neg H \vee \neg I) \wedge (\neg H \vee J) \wedge (\neg H \vee K)$.
- We omit contradictions ($\bot$) in DNF and tautologies ($\top$) in CNF.
  - $(A \vee B) \wedge \neg B \models\!\dashv (A \wedge \neg B) \vee \underbrace{(B \wedge \neg B)}_{\bot} \models\!\dashv A \wedge \neg B$.
  - $A \wedge \underbrace{(B \vee \neg B)}_{\top} \models\!\dashv A$.

Observation 2.13. *By negation of a DNF (and the negation of every its implicant), we get a formula in CNF. However, this formula is a **negation** of the original formula (i.e., not a CNF of the original formula!), and vice versa.*

Example 2.14. Consider the formula in DNF F:

$$(G \wedge \neg H) \vee (\neg I \wedge J \wedge K).$$

Then $\neg F$ is a formula in CNF which looks like:

$$\begin{aligned} \neg((G \wedge \neg H) \vee (\neg I \wedge J \wedge K)) &\mathrel{\text{⫤}\!\!\vDash} \neg(G \wedge \neg H) \wedge \neg(\neg I \wedge J \wedge K) \mathrel{\text{⫤}\!\!\vDash} \\ &\mathrel{\text{⫤}\!\!\vDash} (\neg G \vee H) \wedge (I \vee \neg J \vee \neg K). \end{aligned}$$

However, the CNF of F is obtained by distributing the two implicants, i.e.,

$$\begin{aligned} (G \wedge \neg H) \vee (\neg I \wedge J \wedge K) &\mathrel{\text{⫤}\!\!\vDash} (G \vee (\neg I \wedge J \wedge K)) \wedge (\neg H \vee (\neg I \wedge J \wedge K)) \\ &\mathrel{\text{⫤}\!\!\vDash} ((G \vee \neg I) \wedge (G \vee J) \wedge (G \vee K)) \wedge ((\neg H \vee \neg I) \wedge (\neg H \vee J) \wedge (\neg H \vee K)) \\ &\mathrel{\text{⫤}\!\!\vDash} (G \vee \neg I) \wedge (G \vee J) \wedge (G \vee K) \wedge (\neg H \vee \neg I) \wedge (\neg H \vee J) \wedge (\neg H \vee K) \\ &\not\mathrel{\text{⫤}\!\!\vDash} (\neg G \vee H) \wedge (I \vee \neg J \vee \neg K). \end{aligned}$$

The CNF of F is $(G \vee \neg I) \wedge (G \vee J) \wedge (G \vee K) \wedge (\neg H \vee \neg I) \wedge (\neg H \vee J) \wedge (\neg H \vee K)$ but not $(\neg G \vee H) \wedge (I \vee \neg J \vee \neg K)$ which is **only** a $\neg$ DNF of F.

!!! Remember: $\neg$ DNF **is** CNF, but of a different formula!

2.1.4 Full DNF and CNF

Definition 2.15 (Full DNF and CNF)**.**

- A **minterm** is an implicant which contains all elementary formulas.
- A **maxterm** is a clause which contains all elementary formulas.
- A formula is in **full DNF** if it is a disjunction of minterms.
- A formula is in **full CNF** if it is a conjunction of maxterms.

Theorem 2.16. *For every formula there is a logically equivalent formula which is in full DNF and a logically equivalent formula which is in full CNF.*

Idea for finding full DNF and full CNF:

- Full DNF: complete implicants in DNF to minterms (one by one, adding the missing literals)

$$A \mathrel{\text{⫤}\!\!\vDash} A \wedge \top \mathrel{\text{⫤}\!\!\vDash} A \wedge (B \vee \neg B) \mathrel{\text{⫤}\!\!\vDash} (A \wedge B) \vee (A \wedge \neg B).$$

- Full CNF: complete clauses in CNF to maxterms (one by one, adding the missing literals)

$$A \mathrel{\text{⫤}\!\!\vDash} A \vee \bot \mathrel{\text{⫤}\!\!\vDash} A \vee (B \wedge \neg B) \mathrel{\text{⫤}\!\!\vDash} (A \vee B) \wedge (A \vee \neg B).$$

2.1.5 Logical Consequence

Definition 2.17. A formula B is a **logical consequence** of a formula A – or A logically implies B – if for each valuation v, if $v(A) = 1$ then $v(B) = 1$. We denote it by $A \vDash B$.

Proposition 2.18. *For every two formulas of propositional logic A, B:*

1. $A \mathrel{\text{⫤}\!\!\vDash} B$ *if and only if* $A \vDash B$ *and* $B \vDash A$.
2. $A \vDash B$ *if and only if* $A \Rightarrow B$ *is a tautology.*
3. $A \mathrel{\text{⫤}\!\!\vDash} B$ *if and only if* $A \Leftrightarrow B$ *is a tautology.*
4. $A \vDash B$ *if and only if* $A \wedge \neg B$ *is a contradiction.*

2.2 Exercises

2.2.1 Logical laws

Exercise 2.1. Select some of the laws in Theorem 2.1 and prove their correctness.
Find a formula with at least two elementary formulas which is a tautology. Do the same for a contradiction.

Exercise 2.2. Find logically equivalent formulas which have negation only in front of elementary formulas A, B, C, D.

a) $\neg(A \Rightarrow (B \Rightarrow C))$,

b) $\neg(A \Leftrightarrow (B \wedge (C \Rightarrow D)))$,

c) $\neg(A \vee (B \Rightarrow (C \wedge D)))$,

d) $\neg((A \Rightarrow B) \wedge (C \Leftrightarrow D))$.

Exercise 2.3. Translate the following statements into formulas of PL. Negate the formulas and then use the known laws to "push" the negations in front of elementary formulas. Convert the resulting formulas back into natural language.

a) "If the sun is shining I'll go on a trip or swimming."

b) "Number x is divisible by 6 if and only if it is divisible by 3 and 2."

c) "Bolzano-Cauchy criterion is a necessary condition for a sequence to converge."

Exercise 2.4. Using logical laws simplify the following formulas. Identify all rules you use.

a) $A \Rightarrow (B \vee A)$,

b) $A \Rightarrow (B \Rightarrow (B \Rightarrow A))$,

c) $(A \wedge B) \Rightarrow (A \vee C)$,

d) $(A \Rightarrow B) \vee (B \Rightarrow A)$,

e) $\neg(A \Rightarrow B) \Rightarrow A$,

f) * $\neg((A \Leftrightarrow \neg(B \wedge C)) \wedge A)$.

Exercise 2.5. Pick from the sentences listed below those with the same meaning as "It's not the case that I live in Peru or in Brazil".

a) "I don't live in Peru and I don't live in Brazil."

b) "I don't live in Peru or I don't live in Brazil."

c) "It's not the case that I live in Peru and it's not the case that I live in Brazil."

d) "I don't live in Peru and I live in Brazil."

Exercise 2.6. Pick from the sentences listed below those with the same meaning as "It's not the case that I live in Peru and in Brazil".

a) ”I don’t live in Peru and I don’t live in Brazil.”

b) ”I don’t live in Peru or I don’t live in Brazil.”

c) ”It’s not the case that I live in Peru or it’s not the case that I live in Brazil.”

d) ”I live in Peru and in Brazil.”

2.2.2 DNF and CNF

Exercise 2.7. Convert the following formulas to DNF and to CNF.

a) $A \Rightarrow (B \vee A)$,

b) * $(A \Rightarrow B) \Leftrightarrow (B \Rightarrow A)$,

c) * $\neg((A \Leftrightarrow \neg(B \wedge C)) \wedge A)$,

d) $\neg A \Leftrightarrow (B \vee A)$,

e) $(\neg A \wedge B) \Rightarrow (\neg B \vee C)$,

f) $\neg(A \Rightarrow B) \Rightarrow (\neg A \vee C)$.

Exercise 2.8. Find full DNF and full CNF of the formulas from Exercise 2.7.

Exercise 2.9. Find DNF, CNF, full DNF and full CNF of the following formulas and determine in how many valuations they are true.

a) $(A \Rightarrow B) \wedge B$,

b) $A \Rightarrow (A \wedge B)$,

c) $(A \Rightarrow B) \vee \neg(C \Rightarrow D)$,

d) $\neg((P \wedge \neg Q) \Rightarrow \neg(R \vee S))$,

e) $(A \wedge \neg(C \Rightarrow D)) \Rightarrow (B \wedge E)$,

f) * $(A \wedge B) \Leftrightarrow (A \Rightarrow C)$.

2.2.3 Logical Consequence

Exercise 2.10. Which of the following formulas are logical consequences of $C \wedge D$?

a) C,

b) $C \Rightarrow D$,

c) $D \Rightarrow \neg C$,

d) $\neg C \Rightarrow D$,

e) $C \Leftrightarrow D$,

f) $C \vee \neg D$,

g) $C \wedge \neg D$,

h) $C \vee \neg C$.

Exercise 2.11. Which of the sentences below logically follow from the statement:
"I will take the tram but I will not take the subway." $(T \wedge \neg S)$

1. "If I take the subway then I will take the tram."
2. "I will not take the tram if and only if I will take the subway."
3. "If I take the tram then I will take the subway."
4. "I will not take the tram or I will not take the subway."
5. "I will take the tram or I will not take the tram."

Exercise 2.12. Find and justify logical consequences between pairs of formulas:

a) $\bot,\ \top,\ A,\ B,\ \neg A,\ \neg B,\ A \wedge B,\ A \vee B,\ A \Rightarrow B,\ A \Leftrightarrow B$;

b) $A \Leftrightarrow B,\ A \Rightarrow B,\ B \Rightarrow A,\ A \wedge B,\ \neg A \wedge \neg B,\ A \vee \neg A,\ B \wedge \neg B$.

Exercise 2.13. Convert the formulas below into (full) DNF/CNF and use these forms to decide logical consequence

$$E : \neg(A \Rightarrow (B \Rightarrow C)), \qquad F : \neg((A \Rightarrow B) \Rightarrow C).$$

Exercise 2.14. Convert the formulas below into (full) DNF/CNF and use these forms to decide logical consequence.

$$G : (A \vee B) \Rightarrow C, \qquad H : (A \wedge B) \Rightarrow C, \qquad I : \neg((B \vee C) \Rightarrow (A \wedge B)).$$

2.3 More exercises

Exercise 2.15 (Necessary / sufficient conditions)**.** Formalize the statements below.

a) ”To get to university it is necessary to finish high school.”

b) ”To get to university it is sufficient to finish high school.”

c) ”To get to university it is necessary and sufficient to finish high school.”

Exercise 2.16 (Necessary / sufficient conditions)**.** Formalize the statements below.

a) ”To be able to drive a car it is necessary to be at least 18 years old.”

b) ”To be able to drive a car it is sufficient to be at least 18 years old.”

c) ”To be able to drive a car it is necessary and sufficient to be at least 18 years old.”

Exercise 2.17. Use distributive laws on these formulas.

a) $(A \vee B) \wedge C \wedge D$,

b) $(A \wedge B \wedge C) \vee D$,

c) $A \wedge \neg B \wedge (A \vee B)$.

Exercise 2.18. Use logical laws to show that:

a) $(A \vee B) \wedge (A \vee \neg B) \Dashv\vDash A$;

b) $A \wedge (\neg A \vee B) \Dashv\vDash A \wedge B$;

c) $A \Rightarrow (B \Rightarrow (\neg A \Rightarrow \neg B)) \Dashv\vDash \top$;

d) $(A \Rightarrow B) \Rightarrow (\neg A \Rightarrow \neg B) \Dashv\vDash A \vee \neg B$;

e) $(A \vee B) \Rightarrow (A \wedge B) \Dashv\vDash A \Leftrightarrow B$;

f) $(A \wedge B) \Rightarrow (A \vee B) \Dashv\vDash \top$.

Exercise 2.19. Convert the following formulas to logically equivalent ones containing only the given set of connectives.

a) $(A \Rightarrow B) \wedge C$, negation and disjunction;

b) $(A \vee B) \wedge C$, negation and implication;

c) $A \Rightarrow (B \vee C)$, negation and conjugacy;

d) $(B \vee C) \Rightarrow A$, negation and conjunction.

Exercise 2.20. Pick from the sentences listed below those with the same meaning as ”It’s not the case that if I live in Peru then I live in Brazil”.

a) ”It’s not the case that I live in Peru or it’s not the case that I live in Brazil.”

b) ”I live in Peru and I don’t live in Brazil.”

c) "I live in Peru and in Brazil."

d) "I live in Peru or I don't live in Brazil."

Exercise 2.21. Pick from the sentences listed below those with the same meaning as "It's not the case that I am hungry or it's not the case that I am thirsty".

a) "It's not the case that I am hungry or thirsty".

b) "It's not the case that I am hungry and thirsty".

c) "It's not the case that if I am hungry then I am thirsty".

Exercise 2.22. Pick from the sentences listed below those with the same meaning as "It's not the case that I am hungry and it's not the case that I am thirsty".

a) "It's not the case that I am hungry or thirsty".

b) "It's not the case that I am hungry and thirsty".

c) "It's not the case that if I am hungry then I am thirsty".

Exercise 2.23. * Pick from the sentences listed below those with the same meaning as "I am hungry but it's not the case that I am thirsty".

a) "It's not the case that I am hungry or thirsty".

b) "It's not the case that I am hungry and thirsty".

c) "It's not the case that if I am hungry then I am thirsty".

d) "It's not the case that if I am hungry then I am not thirsty".

Exercise 2.24. Decide which of the following sets are functionally complete.

a) $\{\neg, \wedge\}$,

b) $\{\neg, \Rightarrow\}$,

c) $\{\neg, \Leftrightarrow\}$,

d) $\{\wedge, \vee, \Rightarrow\}$,

e) Peirce symbol NOR: $(A \downarrow B) \Leftrightarrow \neg(A \vee B)$.

Exercise 2.25. Is $\downarrow$ (resp., $\uparrow$) commutative / associative / idempotent?

Exercise 2.26 (Peirce simbol)**.**

1. Express $A \wedge \neg A$ using only $\downarrow$.
2. Express $A \vee \neg A$ using $\downarrow$.

Exercise 2.27. What are the formulas below logically equivalent to (choose from $A, \neg A, \top, \bot$)?

a) $A \downarrow \top$,

b) $A \downarrow \bot$,

c) $A \uparrow \top$,

d) $A \uparrow \bot$.

Exercise 2.28. Find DNF and CNF of the following formulas and determine in how many valuations they are true. Find full DNF and full CNF.

a) $(A \Rightarrow B) \wedge B$,

b) $A \Rightarrow (A \wedge B)$,

c) $(A \Rightarrow B) \vee \neg(C \Rightarrow D)$,

d) $\neg((P \wedge \neg Q) \Rightarrow \neg(R \vee S))$,

e) $(A \wedge \neg(C \Rightarrow D)) \Rightarrow (B \wedge E)$,

f) * $(A \wedge B) \Leftrightarrow (A \Rightarrow C)$.

Exercise 2.29. Convert the formulas below into (full) DNF/CNF and use these forms to decide logical consequence.

$$\begin{aligned} F_1 &: \quad \neg(A \Rightarrow (C \Rightarrow (B \Rightarrow D))), \\ F_2 &: \quad \neg((A \Rightarrow C) \Rightarrow (B \Rightarrow D)), \\ F_3 &: \quad \neg(A \Rightarrow (C \Rightarrow (B \Rightarrow (D \Rightarrow A)))). \end{aligned}$$

Remark 2.19. The formula F_3 above is a contradiction. That is the only formula for which no full DNF exists. We can reason this out as follows: each minterm in full DNF of a formula A corresponds precisely to one truth valuation v for which $v(A) = 1$ and vice versa, each truth valuation v such that $v(A) = 1$ corresponds to precisely one minterm in full DNF of A. Since $\bot$ is true for no truth valuation there can be no minterm corresponding to it. We just write $\bot$.

Exercise 2.30. Convert the formulas below into (full) DNF/CNF and use these forms to decide logical consequence.

$$E : (A \wedge B) \Rightarrow (C \wedge D), \qquad F : A \Rightarrow (B \Rightarrow (C \Rightarrow D)).$$

Exercise 2.31. ** Verify that $A \Rightarrow \bot \mathrel{\models\!\!\!\mid} \neg A$. Use this fact to prove that $\{\bot, \Rightarrow\}$ represents a functionally complete system.